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SPECTRAL INCLUSION FOR UNBOUNDED DIAGONALLY DOMINANT $n \times n$ OPERATOR MATRICES

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ABSTRACT. In this paper, we establish an analytic enclosure for the spectrum of unbounded linear operators $\mathcal{A}$ admitting an $n \times n$ matrix representation in a Hilbert space $\mathcal{H} = \mathcal{H}_1 \oplus \cdots \oplus \mathcal{H}_n$. For diagonally dominant operator matrices of order 0, we show that this new enclosing set, the block numerical range $W^n(\mathcal{A})$, contains the eigenvalues of $\mathcal{A}$ and that the approximate point spectrum of $\mathcal{A}$ is contained in its closure $\overline{W^n(\mathcal{A})}$. Since the block numerical range turns out to be a subset of the usual numerical range, $W^n(\mathcal{A}) \subset W(\mathcal{A})$, it may give a tighter enclosure of the spectrum. Moreover, we prove Gershgorin theorems for diagonally dominant $n \times n$ operator matrices and compare our results to both Gershgorin bounds and classical perturbation theory. Our results are illustrated by deriving new lower bounds for 3×3 self-adjoint operator matrices and applying the latter to three-channel Hamiltonians in quantum mechanics.

1. Introduction. The location of spectra of non-self-adjoint linear operators plays a crucial role in many applications. Especially for non-self-adjoint operators but also for self-adjoint operators with spectral gaps, numerical approximations of the spectrum or of eigenvalues are prone to be unreliable (see, e.g., [**2**]). On the other hand, rigorous analytic information on the spectrum is, in general, difficult to obtain (see, e.g., [**5**]). One of the few simple analytical tools for localizing the spectrum is the numerical range; however, due to its convexity, the numerical range is often too coarse to provide good enclosures

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for the spectrum or even useless, e.g., for estimating eigenvalues in spectral gaps.

In this paper, we establish a new analytic enclosure for the spectrum of unbounded linear operators that admit a matrix representation $\mathcal{A} = (A_{ij})_{i,j=1}^n$ with respect to a decomposition $\mathcal{H} = \mathcal{H}_1 \oplus \cdots \oplus \mathcal{H}_n$ of a Hilbert space $\mathcal{H}$. The enclosing set, the *block numerical range* $W^n(\mathcal{A})$, is defined as the union of all eigenvalues of the $n \times n$ matrices

$$\mathcal{A}_f := ((A_{ij} f_j, f_i))_{i,j=1}^n \in M_n(\mathbb{C}), \quad f = (f_i)_{i=1}^n \in D(\mathcal{A}), \ \|f_i\| = 1.$$

One of our main results is the *spectral inclusion property* of the block numerical range: all eigenvalues of $\mathcal{A}$ are contained in $W^n(\mathcal{A})$ and, if $\mathcal{A}$ is diagonally dominant of order 0, the approximate point spectrum of $\mathcal{A}$ is contained in the closure $\overline{W^n(\mathcal{A})}$,

$$\sigma_{\mathrm{p}}(\mathcal{A}) \subset W^n(\mathcal{A}), \qquad \sigma_{\mathrm{app}}(\mathcal{A}) \subset \overline{W^n(\mathcal{A})}. \tag{1.1}$$

This is a direct generalization of the spectral inclusion property of the classical numerical range $W(\mathcal{A}) = W^1(\mathcal{A})$ (see [**9**]) obtained for $n = 1$.

Further new results include criteria for the closedness of tridiagonal $n \times n$ operator matrices which improve an earlier result for the case $n = 2$ in [**14**], inclusions among different block numerical ranges, especially the enclosure $W^n(\mathcal{A}) \subset W(\mathcal{A})$ in the classical numerical range, Gershgorin type theorems for general $n \times n$ operator matrices, and new lower bounds for self-adjoint diagonally dominant 3×3 operator matrices with diagonal entries all bounded from below.

The quadratic numerical range (which is the special case $n = 2$) was first introduced in [**11**] for unbounded operator matrices with *bounded* off-diagonal entries. The block numerical range of *bounded* $n \times n$ operator matrices was introduced, and its spectral inclusion property was proved in [**15**]. For the quadratic numerical range, the spectral inclusion property was proved in [**11**] only in a special case; it was assumed that *all off-diagonal entries are bounded* and the diagonal entries are *separated*, i.e.,

$$\operatorname{Re} \overline{W(A_{11})} < \operatorname{Re} \overline{W(A_{22})}.$$

In [**14**], the spectral inclusion property was generalized to *diagonally dominant and off-diagonally dominant* operator matrices *of order* 0. For $n \geq 3$, up until now, there have been no spectral inclusion results in the unbounded case and, even in the bounded case, no estimates of the block numerical range.

The present paper fills in these gaps and provides new spectral estimates, even for semi-bounded self-adjoint 3×3 operator matrices. Moreover, due to the inclusion

$$W^n(\mathcal{A}) \subset W^{n-1}(\mathcal{A}) \subset \cdots \subset W^2(\mathcal{A}) \subset W^1(\mathcal{A}) = W(\mathcal{A})$$

upon refinement of the decomposition of the Hilbert space $\mathcal{H}$, block numerical ranges may give tighter enclosures than the classical numerical range $W(\mathcal{A})$. Since, unlike $W(\mathcal{A})$, the block numerical range $W^n(\mathcal{A})$ is in general no longer convex, and its at most n components need not be so, our new spectral enclosures may indeed be considerably better.

The paper is organized as follows. In Section 2, we introduce and study the notion of diagonal dominance for $n \times n$ operator matrices with unbounded entries. In Section 3, we establish criteria for the closedness of tridiagonal diagonally dominant operator matrices, thus improving the results for $n = 2$ in **[14]**. In Section 4, we introduce the block numerical range $W^n(\mathcal{A})$ of an unbounded $n \times n$ operator matrix $\mathcal{A}$ and prove some elementary properties of it; in particular, we show that the block numerical range is always contained in the usual numerical range. In Section 5, we prove our main result which shows that the block numerical range has the spectral inclusion property (1.1) if the operator matrix is diagonally dominant of order 0. In Section 6, we establish a Gershgorin type theorem for unbounded $n \times n$ operator matrices and derive an estimate for the block numerical range by means of the matrix Gershgorin theorem. In Section 7, we use the cubic numerical range $W^3(\mathcal{A})$ to establish new estimates for the spectrum of tridiagonal 3×3 self-adjoint operator matrices. We compare our new bounds to classical perturbation theory as well as to the Gershgorin bounds from Section 6, and illustrate them by an application to three-channel Hamiltonians from quantum mechanics.

Throughout this paper, we use the following notation. If $\mathcal{A}$ is a linear operator from one Banach or Hilbert space to another, then $D(\mathcal{A})$ denotes its domain, $R(\mathcal{A})$ its range, $\rho(\mathcal{A})$ its resolvent set, $\sigma(\mathcal{A})$ its spectrum, $\sigma_{\mathrm{p}}(\mathcal{A})$ its point spectrum and $\sigma_{\mathrm{app}}(\mathcal{A})$ its approximate point spectrum.

2. Diagonally dominant $n \times n$ operator matrices. Let $n \in \mathbb{N}$, $n \geq 2$, let $(\mathcal{H}_i, \|\cdot\|_i)$, $i = 1, \dots, n$, be Banach spaces and let $(\mathcal{H}, \|\cdot\|)$ be the Euclidean product of $\mathcal{H}_1, \dots, \mathcal{H}_n$, that is,

$$\begin{aligned} &\mathcal{H} := \mathcal{H}_1 \oplus \cdots \oplus \mathcal{H}_n, \\ &\|f\| := \sqrt{\|f_1\|_1^2 + \cdots + \|f_n\|_n^2}, \quad f = (f_1, \dots, f_n)^t \in \mathcal{H}. \end{aligned} \tag{2.1}$$

In the Banach space $\mathcal{H}$ we consider linear operators $\mathcal{A}$ that admit an $n \times n$ operator matrix representation

$$\mathcal{A} = (A_{ij})_{i,j=1}^n \quad \text{in} \;\; \mathcal{H} = \mathcal{H}_1 \oplus \cdots \oplus \mathcal{H}_n, \tag{2.2}$$

where the entries are densely defined closable linear operators

$$A_{ij} : \mathcal{H}_j \supset D(A_{ij}) \longrightarrow \mathcal{H}_i, \quad i, j = 1, \dots, n,$$

and for which the domain of $\mathcal{A}$, given by

$$D(\mathcal{A}) = \bigoplus_{j=1}^n D_j, \qquad D_j := \bigcap_{i=1}^n D(A_{ij}) \subset \mathcal{H}_j, \tag{2.3}$$

is again dense in $\mathcal{H}$.

For the reader's convenience, we briefly recall the notion of relative boundedness (see [**9,** subsection VI.1]).

Let $(E, \|\cdot\|_E)$, $(F, \|\cdot\|_F)$, $(G, \|\cdot\|_G)$ be Banach spaces, and let

$$T : E \supset D(T) \longrightarrow F, \qquad S : E \supset D(S) \longrightarrow G$$

be linear operators. Then, S is called T-bounded (or relatively bounded with respect to T) if $D(T) \subset D(S)$, and there exist constants a_S, $b_S \geq 0$ with

$$\|Sx\|_G \leq a_S \|x\|_E + b_S \|Tx\|_F, \quad x \in D(T); \tag{2.4}$$

the infimum δ_S of all b_S such that (2.4) holds for some $a_S \geq 0$ is called the T-bound of S (or relative bound of S with respect to T).

Note that, if T is closed and S is closable with $D(T) \subset D(S)$, then S is T-bounded (see [**9,** Remark IV.1.5]). It is also not difficult to prove (see [**9,** subsection V.4.1, (4.1), (4.2)]) that (2.4) is equivalent to

$$\|Sx\|_G^2 \leq a_S'^2 \|x\|_E^2 + b_S'^2 \|Tx\|_F^2, \quad x \in D(T), \tag{2.5}$$

with constants a'_S, $b'_S \geq 0$; moreover, (2.4) holds with $b_S < \delta$ for some $\delta > 0$ if and only if (2.5) holds with some $b'_S < \delta$. Hence, the T-bound δ_S of S can also be defined as the infimum of all $b'_S \geq 0$ so that (2.5) holds for some $a'_S \geq 0$.

Definition 2.1. For an operator matrix $\mathcal{A}$ as in (2.2), we define the *diagonal part* $\mathcal{T}$ and the *off-diagonal part* $\mathcal{S}$ by

$$\mathcal{T} := \operatorname{diag}(A_{11}, \dots, A_{nn}), \qquad \mathcal{S} := \mathcal{A} - \mathcal{T}, \tag{2.6}$$

and we call $\mathcal{A}$ *diagonally dominant of order* $\delta_{\mathcal{S}}$ if $\mathcal{S}$ is $\mathcal{T}$-bounded with $\mathcal{T}$-bound $\delta_{\mathcal{S}}$.

Note that, for a diagonally dominant operator matrix $\mathcal{A}$, the domain is always given by the domains of the diagonal entries

$$D(\mathcal{A}) = D(A_{11}) \oplus \cdots \oplus D(A_{nn}). \tag{2.7}$$

Remark 2.2. If $\mathcal{A}$ is diagonally dominant of order $\delta_{\mathcal{S}} < 1$, then $\mathcal{S}$ is $\mathcal{A}$-bounded with $\mathcal{A}$-bound $\leq \delta_{\mathcal{S}}/(1 - \delta_{\mathcal{S}})$.

Diagonal dominance may also be characterized by means of the entries of the operator matrix $\mathcal{A}$ as follows.

Proposition 2.3. *Let $\mathcal{A}$ be as in* (2.2)*. Then,*

$$A \textit{ is diagonally dominant} \iff A_{ij} \textit{ is } A_{jj}\textit{-bounded}$$

for all $i, j = 1, \dots, n$, $i \neq j$. In this case, if $\delta_{\mathcal{S}}$ is the dominance order of $\mathcal{A}$ and δ_{ij} are the A_{jj}-bounds of A_{ij}, then

$$\delta_{ij} \leq \delta_{\mathcal{S}} \leq \delta := \left((n-1) \max_{l=1}^{n} \sum_{\substack{k=1 \\ k \neq l}}^{n} \delta_{kl}^2 \right)^{1/2} \tag{2.8}$$

for all $i, j = 1, \dots, n$, $i \neq j$; in particular,

$$\mathcal{A} \textit{ is diagonally dominant of order } 0 \iff \delta_{ij} = 0.$$

Proof. To show that $\delta_{ij} \leq \delta_{\mathcal{S}}$, let $j \in \{1, \dots, n\}$, $f_j \in \mathcal{H}_j$, $e_j := (0, \dots, 0, f_j, 0, \dots, 0)^t$ with the only non-zero entry f_j in the jth line, and let $\varepsilon > 0$ be arbitrary. Then, by assumption, there exist $a'_{\mathcal{S}}, b'_{\mathcal{S}} \geq 0$, $\delta_{\mathcal{S}} \leq b'_{\mathcal{S}} \leq \delta_{\mathcal{S}} + \varepsilon$ such that, for every $i \in \{1, \dots, n\}$, $i \neq j$,

$$\begin{aligned}\|A_{ij} f_j\|_i^2 &\leq \sum_{\substack{k=1\\k\neq j}}^n \|A_{kj} f_j\|_k^2 = \|\mathcal{S} e_j\|^2 \\ &\leq a'^2_{\mathcal{S}} \|e_j\|^2 + b'^2_{\mathcal{S}} \|\mathcal{T} e_j\|^2 = a'^2_{\mathcal{S}} \|f_j\|_j^2 + b'^2_{\mathcal{S}} \|A_{jj} f_j\|_j^2.\end{aligned}$$

To show that $\delta_{\mathcal{S}} \leq \delta$, let $\varepsilon > 0$ be arbitrary. By the assumptions, the domain inclusions $D(A_{jj}) \subset D(A_{ij})$ hold for $i, j = 1, \dots, n$, $i \neq j$, and there are constants $a'_{ij}, b'_{ij} \geq 0$ with $\delta_{ij} \leq b'_{ij} < \delta_{ij} + \varepsilon$ and

$$\|A_{ij} f_j\|_i^2 \leq a'^2_{ij} \|f_j\|_j^2 + b'^2_{ij} \|A_{jj} f_j\|_j^2, \quad f_j \in D(A_{jj}).$$

Therefore,

$$D(\mathcal{T}) = D(A_{11}) \oplus \cdots \oplus D(A_{nn}) \subset D(\mathcal{S})$$

and, for $f = (f_1, \dots, f_n)^t \in D(\mathcal{T})$,

(2.9)

$$\begin{aligned}\|\mathcal{S} f\|^2 = \sum_{i=1}^n \Bigg\| \sum_{\substack{j=1\\j\neq i}}^n A_{ij} f_j \Bigg\|_i^2 &\leq (n-1) \sum_{i=1}^n \sum_{\substack{j=1\\j\neq i}}^n \|A_{ij} f_j\|_i^2 \\ &\leq (n-1) \sum_{i=1}^n \sum_{\substack{j=1\\j\neq i}}^n \big(a'^2_{ij} \|f_j\|_j^2 + b'^2_{ij} \|A_{jj} f_j\|_j^2\big) \\ &= (n-1) \sum_{j=1}^n \Bigg(\sum_{\substack{i=1\\i\neq j}}^n a'^2_{ij} \Bigg) \|f_j\|_j^2 + (n-1) \sum_{j=1}^n \Bigg(\sum_{\substack{i=1\\i\neq j}}^n b'^2_{ij} \Bigg) \|A_{jj} f_j\|_j^2 \\ &\leq (n-1) \Bigg(\max_{j=1}^n \sum_{\substack{i=1\\i\neq j}}^n a'^2_{ij} \Bigg) \|f\|^2 + (n-1) \Bigg(\max_{j=1}^n \sum_{\substack{i=1\\i\neq j}}^n b'^2_{ij} \Bigg) \|\mathcal{T} f\|^2.\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary and $\delta_{ij} \leq b'_{ij} < \delta_{ij} + \varepsilon$, we obtain that $\delta_{\mathcal{S}} \leq \delta$.

The last equivalence is immediate from the two-sided estimate in (2.8). □

Remark 2.4. If the operator matrix $\mathcal{A}$ has a fixed number of zero entries in each column, then the upper bound δ in Proposition 2.3 may be improved. For example, if $\mathcal{A}$ is tridiagonal, $A_{ij} = 0$ for $|i-j| > 1$, we can replace the estimate in the first line of (2.9) by the equality

$$\|\mathcal{S}f\|^2 = \|A_{12}f_2\|_1^2 + \sum_{i=2}^{n-1} \|A_{i,i-1}f_{i-1} + A_{i,i+1}f_{i+1}\|_i^2 + \|A_{n,n-1}f_{n-1}\|_n^2$$

to obtain that

$$\delta_{\mathcal{S}} \le \widehat{\delta} := \max\Big\{\sqrt{2}\delta_{21}, \sqrt{\delta_{12}^2 + 2\delta_{32}^2}, \sqrt{2}\max_{j=3}^{n-2}(\delta_{j-1,j}^2 + \delta_{j+1,j}^2), \\ \sqrt{\delta_{n,n-1}^2 + 2\delta_{n-2,n-1}^2}, \sqrt{2}\delta_{n-1,n}\Big\}.$$

Corollary 2.5. *If the diagonal entries A_{jj} of $\mathcal{A}$ are closed, then*

$$\mathcal{A} \text{ is diagonally dominant} \iff D(A_{jj}) \subset D(A_{ij})$$

for all $i, j = 1, \ldots, n$, $i \neq j$.

Proof. The implication $\Rightarrow$ is obvious since, by assumption,

$$\bigoplus_{j=1}^{n} D(A_{jj}) = D(\mathcal{T}) \subset D(\mathcal{S}) = \bigoplus_{j=1}^{n}\Bigg(\bigcap_{\substack{i=1\\ i\neq j}}^{n} D(A_{ij})\Bigg).$$

Since A_{jj} is closed and A_{ij} is closable for $i, j = 1, \ldots, n$, $i \neq j$, the inclusion $D(A_{jj}) \subset D(A_{ij})$ implies that A_{ij} is A_{jj}-bounded (see **[9,** Remark IV.1.5]). Now, Proposition 2.3 implies the implication $\Leftarrow$. $\square$

In the last part of the proof, we cannot directly conclude that the domain inclusions on the entries imply $D(\mathcal{S}) \subset D(\mathcal{T})$ and hence the claim on the left hand side. The last conclusion only holds if $\mathcal{S}$ were closable, which need not be the case if $n \geq 3$.

Remark 2.6. The diagonal part $\mathcal{T}$ of $\mathcal{A}$ is always closable since it is the direct sum of the closable operators $A_{11}, \ldots, A_{nn}$. For the off-diagonal part $\mathcal{S}$ this is only true in the 2×2 case.

Indeed, if $n = 2$, A_{12}, A_{21} are closable, and $((y_\nu, w_\nu)^{\mathrm{t}})_1^\infty \subset \mathcal{H}_1 \oplus \mathcal{H}_2$ is such that

$$\begin{pmatrix} y_\nu \\ w_\nu \end{pmatrix} \longrightarrow \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \qquad \mathcal{S}\begin{pmatrix} y_\nu \\ w_\nu \end{pmatrix} = \begin{pmatrix} A_{12}w_\nu \\ A_{21}y_\nu \end{pmatrix} \longrightarrow \begin{pmatrix} y \\ w \end{pmatrix}, \quad \nu \to \infty,$$

then $y = 0$ and $w = 0$ since A_{12} and A_{21}, respectively, are closable.

On the other hand, if $n \geq 3$, then $\mathcal{S}$ need not be closable even if $\mathcal{A}$ is tridiagonal and all entries are closed. As an example, consider the case $n = 3$ and

$$\mathcal{S} = \begin{pmatrix} 0 & A_{12} & 0 \\ A_{21} & 0 & A_{23} \\ 0 & A_{32} & 0 \end{pmatrix}$$

with densely defined closed entries

$$A_{ij} : \mathcal{H}_j \supset D(A_{ij}) \longrightarrow \mathcal{H}_i, \quad i, j = 1, 2, 3, \quad |i - j| = 1.$$

Suppose that $\mathcal{H}_1 = \mathcal{H}_3$, $A_{21} = A_{21}^0 + A_{21}^1$, $A_{23} = -A_{21}^0$ such that A_{21}^0 is closed and $A_{21}^1 = A_{21} + A_{23}$ is A_{21}^0-bounded with A_{21}^0-bound < 1, but not closable. Then there exists a sequence $(y_\nu)_1^\infty \subset \mathcal{H}_1 = \mathcal{H}_3$ such that $y_\nu \longrightarrow 0$, $(A_{21} + A_{23})y_\nu \longrightarrow v \neq 0$ for $\nu \to \infty$. Then, $\mathcal{S}$ is not closable since, for the sequence $((y_\nu, 0, y_\nu)^t)_1^\infty \subset \mathcal{H}$, we have

$$\begin{pmatrix} y_\nu \\ 0 \\ y_\nu \end{pmatrix} \longrightarrow \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \quad \mathcal{S}\begin{pmatrix} y_\nu \\ 0 \\ y_\nu \end{pmatrix} = \begin{pmatrix} 0 \\ (A_{21} + A_{23})y_\nu \\ 0 \end{pmatrix} \longrightarrow \begin{pmatrix} 0 \\ v \\ 0 \end{pmatrix}, \quad \nu \to \infty.$$

For example, one could choose $\mathcal{H}_1 = \mathcal{H}_2 = \mathcal{H}_3 = L_2(0, 1)$, $A_{21}^0 y := -y''$, $A_{21}^1 = y'(0) \cdot 1$ with $D(A_{21}^0) = D(A_{21}^1) = W_2^2(0, 1)$; then, A_{21}^0 is closed and A_{21}^1 is A_{21}^0-bounded with A_{21}^0-bound 0, but not closable (see [**1**]).

The upper bounds δ and $\widehat{\delta}$ for the dominance order in Proposition 2.3 and Remark 2.4, respectively, may be strict, as the next example shows.

Example 2.7. Let $n = 3$, let the Hilbert spaces $\mathcal{H}_1 = \mathcal{H}_2 = \mathcal{H}_3$ coincide, and suppose that $\mathcal{H}_2 = \mathcal{H}_2^1 \oplus \mathcal{H}_2^2$ where $(\mathcal{H}_2^i, \|\cdot\|_{2,i})$, $i = 1, 2$, are non-trivial invariant subspaces for $A_{22} : \mathcal{H}_2 \supset D(A_{22}) \longrightarrow \mathcal{H}_2$. Denote by $P_i : \mathcal{H}_2 \longrightarrow \mathcal{H}_2^i$ the projection from $\mathcal{H}_2$ to $\mathcal{H}_2^i$, $i = 1, 2$, and let

$\gamma > 0$. Then the tridiagonal operator matrix $\mathcal{A}$

$$\mathcal{A} := \begin{pmatrix} 0 & A_{12} & 0 \\ 0 & A_{22} & 0 \\ 0 & A_{32} & 0 \end{pmatrix}, \qquad A_{12} := \begin{pmatrix} \gamma P_1 A_{22} P_1^* & 0 \\ 0 & 0 \end{pmatrix},$$

$$A_{32} := \begin{pmatrix} 0 & 0 \\ 0 & \gamma P_2 A_{22} P_2^* \end{pmatrix},$$

is diagonally dominant of order $\delta_{\mathcal{S}} = \gamma$ since here $D(\mathcal{T}) = D(\mathcal{S}) = \mathcal{H}_1 \oplus D(A_{22}) \oplus \mathcal{H}_3$ and, for $f = (f_1, f_2, f_3)^t \in D(\mathcal{T})$ and $f_2 = (f_2^1, f_2^2)^t \in P_1 D(A_{22}) \oplus P_2 D(A_{22})$,

$$\begin{aligned} \|\mathcal{S} f\|^2 &= \|A_{12} f_2\|_1^2 + \|A_{32} f_2\|_3^2 = \|\gamma P_1 A_{22} f_2\|_{2,1}^2 + \|\gamma P_2 A_{22} f_2\|_{2,2}^2 \\ &= \gamma \|A_{22} f_2\|_2^2 = \gamma \|\mathcal{T} f\|^2. \end{aligned}$$

On the other hand, since A_{12} and A_{32} are both A_{22}-bounded with A_{22}-bounds $\delta_{12} = \delta_{32} = \gamma$, the bound $\widehat{\delta} = \sqrt{\delta_{12}^2 + 2\delta_{32}^2} = \sqrt{3}\gamma$ in Remark 2.4 is strictly greater than $\delta_{\mathcal{S}} = \gamma$ (and the bound $\delta = \sqrt{2}\sqrt{\delta_{12}^2 + \delta_{32}^2} = 2\gamma$ is even greater).

Remark 2.8. Note that the dominance order $\delta_{\mathcal{S}}$ depends upon the choice of the norm on $\mathcal{H}$ since so do the constants $a_{\mathcal{S}}$ and $b_{\mathcal{S}}$ in (2.4). For instance, instead of the Euclidean norm $\|\cdot\|$ on the Hilbert space product $\mathcal{H} = \mathcal{H}_1 \oplus \cdots \oplus \mathcal{H}_n$, choose the equivalent norm

$$|||f|||_1 := \|f_1\|_1 + \cdots + \|f_n\|_n, \qquad f = (f_1, \ldots, f_n)^t \in \mathcal{H},$$

and let $\delta'_{\mathcal{S}}$ be the corresponding dominance order, and δ_{ij} the A_{ii}-bound of A_{ij}, $i, j = 1, \ldots, n$, $|i - j| = 1$. Then, we obtain the upper bound

$$\delta'_{\mathcal{S}} \le \delta' := \max_{j=1}^{n} \sum_{\substack{i=1 \\ i \ne j}}^{n} \delta_{ij}.$$

Note that, in the tridiagonal case, no further improvement may be obtained; here $\delta_{ij} = 0$ for $|i - j| > 1$, whence

$$\delta' = \widehat{\delta}' = \max\left\{ \delta_{21}, \max_{j=2}^{n-1} (\delta_{j-1,j} + \delta_{j+1,j}), \delta_{n-1,n} \right\}.$$

3. Closability/closedness of diagonally dominant $n \times n$ operator matrices. Next, we establish conditions for the closability and closedness of general $n \times n$ operator matrices and for the more particular tridiagonal case. The first result is a simple perturbation result.

Theorem 3.1. *If the operator matrix $\mathcal{A}$ in* (2.2) *is diagonally dominant of order $\delta_{\mathcal{S}} < 1$, then $\mathcal{A}$ is closable*; *if, in addition, the diagonal entries of $\mathcal{A}$ are closed, then $\mathcal{A}$ is closed.*

Proof. By assumption, $\mathcal{A} = \mathcal{T} + \mathcal{S}$ with $\mathcal{T}$ closable or closed, respectively, and $\mathcal{S}$ is $\mathcal{T}$-bounded with $\mathcal{T}$-bound $\delta_{\mathcal{S}} < 1$. Now, both assertions follow from classical perturbation results on the stability of closability and closedness, respectively (see [**9,** Theorem IV.1.1]). □

In the following, we derive another criterion for the closability and closedness, respectively, of *tridiagonal* operator matrices which are characterized by $A_{ij} = 0$ for $|i - j| > 1$, i.e.,

$$\mathcal{A} := \begin{pmatrix} A_{11} & A_{12} & 0 & \cdots & 0 & 0 & 0 \\ A_{21} & A_{22} & A_{23} & & & 0 & 0 \\ 0 & A_{32} & A_{33} & \ddots & & & 0 \\ \vdots & & \ddots & \ddots & \ddots & & \vdots \\ 0 & & & \ddots & \ddots & A_{n-2,n-1} & 0 \\ 0 & 0 & & & A_{n-1,n-2} & A_{n-1,n-1} & A_{n-1,n} \\ 0 & 0 & 0 & \cdots & 0 & A_{n,n-1} & A_{nn} \end{pmatrix}; \tag{3.1}$$

here, the relative bounds $\delta_{j+1,j}$ and $\delta_{j,j+1}$ of $A_{j+1,j}$ and $A_{j,j+1}$ on the lower and upper off-diagonal, respectively, may balance each other.

The next theorem generalizes, and improves, the closability/closedness criterion for $n = 2$ in [**13,** Theorem 2.2.8].

Theorem 3.2. *Let $\mathcal{A} = (A_{ij})_{i,j=1}^{n}$ be tridiagonal and diagonally dominant and let δ_{ij} be the A_{ii}-bound of A_{ij}, $i, j = 1, \dots, n$, $|i - j| = 1$. If*

$$\delta_n(\mathcal{A}) = \cfrac{\delta_{n,n-1}\delta_{n-1,n}}{1 - \cfrac{\delta_{n-1,n-2}\delta_{n-2,n-1}}{1 - \cfrac{\delta_{n-2,n-3}\delta_{n-3,n-2}}{1 - \cfrac{\ddots}{\ddots \quad 1 - \cfrac{\delta_{32}\delta_{23}}{1-\delta_{21}\delta_{12}}}}}} < 1, \tag{3.2}$$

then $\mathcal{A}$ is closable and closed if its diagonal elements A_{ii} are closed.

Remark 3.3. The continued fraction $\delta_n(\mathcal{A})$ in (3.2) can also be defined by the recursion

$$\delta_1(\mathcal{A}) := 0, \qquad \delta_k(\mathcal{A}) := \frac{\delta_{k,k-1}\delta_{k-1,k}}{1-\delta_{k-1}(\mathcal{A})}, \quad k = 2, 3, \ldots, n. \tag{3.3}$$

Then, condition (3.2) is equivalent to

$$\delta_{k,k-1}\delta_{k-1,k} + \delta_{k-1}(\mathcal{A}) < 1, \quad k = 2, 3, \ldots, n; \tag{3.4}$$

this shows that (3.2) necessitates that the relative bounds satisfy $\delta_{k,k-1}\delta_{k-1,k} < 1$, $k = 2, 3, \ldots, n$.

Proof of Theorem 3.2. For $\mu > 0$, define the $n \times n$ diagonal operator matrix as

$$\mathcal{M}(\mu) := \operatorname{diag}(I_{\mathcal{H}_1}\ \mu I_{\mathcal{H}_2}\ \mu^2 I_{\mathcal{H}_3} \cdots \mu^{n-1} I_{\mathcal{H}_n}).$$

Clearly, $\mathcal{A}$ is closable or closed if and only if, for some $\mu > 0$, so is

$$\begin{aligned}\mathcal{A}_{nn}(\mu) &:= \mathcal{M}(\mu)\mathcal{A}\mathcal{M}(\mu)^{-1}\\ &= \begin{pmatrix} A_{11} & \frac{1}{\mu}A_{12} & 0 & \cdots & \cdots & 0 \\ \mu A_{21} & A_{22} & \frac{1}{\mu}A_{23} & & & 0 \\ 0 & \mu A_{32} & A_{33} & \frac{1}{\mu}A_{34} & & 0 \\ \vdots & & \ddots & \ddots & \ddots & \vdots \\ \vdots & & & \ddots & \ddots & \frac{1}{\mu}A_{n-1,n} \\ 0 & \cdots & \cdots & \cdots & \mu A_{n,n-1} & A_{nn}. \end{pmatrix}.\end{aligned}$$

The latter will be proved by induction on $n = 2, 3, \ldots$. In order to formulate the precise induction claim, we introduce the following temporary notation.

For $k = 1, 2, \ldots, n$, we denote by $\widehat{\mathcal{H}}_k := \mathcal{H}_1 \oplus \cdots \oplus \mathcal{H}_k$ the first k components of $\mathcal{H}$ and by $\widehat{\mathcal{H}}_{n-k}$ the last $n-k$ components of $\mathcal{H}$; the elements of $\widetilde{H}_k$ are denoted by $\widetilde{f}_k = (f_i)_{i=1}^k$ with $f_i \in \mathcal{H}_i$, $i = 1, 2, \ldots, k$. Then, $\mathcal{H} = \widetilde{\mathcal{H}}_k \oplus \widehat{\mathcal{H}}_{n-k}$ for $k = 1, 2, \ldots, n-1$ and $\mathcal{H} = \widetilde{\mathcal{H}}_n$ for $k = n$. We denote by $P_k : \mathcal{H} \longrightarrow \widetilde{\mathcal{H}}_k$ the projection of $\mathcal{H}$ onto the first k components, by $Q_{n-k} : \mathcal{H} \longrightarrow \widehat{\mathcal{H}}_{n-k}$ the projection of $\mathcal{H}$ onto the last $n-k$ components, and we set $\mathcal{A}_{kk} := P_k \mathcal{A} P_k^*$ for $k = 1, 2, \ldots, n$. Then, with respect to the decomposition $\widetilde{\mathcal{H}}_n = \widetilde{\mathcal{H}}_{n-1} \oplus \mathcal{H}_n$, the $n \times n$ operator matrix $\mathcal{A}_{nn}(\mu)$ has the 2×2 operator matrix representation

$$\mathcal{A}_{nn}(\mu) = \begin{pmatrix} \mathcal{A}_{n-1,n-1}(\mu) & \mathcal{A}_{n-1,n}(\mu) \\ \mathcal{A}_{n,n-1}(\mu) & \mathcal{A}_{nn} \end{pmatrix},$$

where

$$\begin{aligned} \mathcal{A}_{n-1,n}(\mu) &:= P_{n-1} \mathcal{A}_{nn}(\mu) Q_1^* = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \frac{1}{\mu} \mathcal{A}_{n-1,n} \end{pmatrix}, \\ \mathcal{A}_{n,n-1}(\mu) &:= Q_1 \mathcal{A}_{nn}(\mu) P_{n-1}^* = \begin{pmatrix} 0 & \cdots & 0 & \mu \mathcal{A}_{n,n-1} \end{pmatrix}. \end{aligned}$$

Further, for $\mu \geq 0$, we define $\delta_1(\mathcal{A}, \mu) := 0$,

$$\delta_k(\mathcal{A}, \mu) := \delta_{k,k-1} \sqrt{\mu^2 + \delta_{k-1,k}^2} \, \frac{1}{1 - \delta_{k-1}(\mathcal{A}, \mu)}, \quad k = 2, 3, \ldots, n.$$

Note that, by definition (3.3) and induction, we have $\delta_k(\mathcal{A}, 0) = \delta_k(\mathcal{A})$, $k = 1, 2, \ldots, n$.

By induction on $n = 2, 3, \ldots$, we prove that there exists a $\mu_n > 0$ such that, for $\mu \in (0, \mu_n)$,

(i)$_n$ $\delta_{n-1}(\mathcal{A}, \mu) < 1$ and $\mathcal{A}_{n,n-1}(\mu)$ is $\mathcal{A}_{n-1,n-1}(\mu)$-bounded with $\mathcal{A}_{n-1,n-1}(\mu)$-bound

$$\widehat{\delta}_n(\mu) \leq \mu \, \delta_{n,n-1} \frac{1}{1 - \delta_{n-1}(\mathcal{A}, \mu)}; \tag{3.5}$$

$(\text{ii})_n$ $\mathcal{A}_{nn}(\mu)$ is closable and closed if all its diagonal elements $A_{11}, \dots, A_{nn}$ are closed.

First, let $n = 2$.

$(\text{i})_2$ Since $\mathcal{A}_{21}(\mu) = \mu A_{21}$ and $\mathcal{A}_{11}(\mu) = A_{11}$, it follows that $\mathcal{A}_{21}(\mu)$ is $\mathcal{A}_{11}(\mu)$-bounded with $\mathcal{A}_{11}(\mu)$-bound

$$\mu\delta_{21} = \mu\delta_{21}\frac{1}{1 - \delta_1(\mathcal{A}, \mu))}$$

for every $\mu > 0$ since $\delta_1(\mathcal{A}, \mu) = 0$ by definition.

$(\text{ii})_2$ The relative boundedness constants for

$$\mathcal{A}_{22}(\mu) = \begin{pmatrix} I_{\mathcal{H}_1} & 0 \\ 0 & \mu I_{\mathcal{H}_2} \end{pmatrix} \mathcal{A} \begin{pmatrix} I_{\mathcal{H}_1} & 0 \\ 0 & \frac{1}{\mu} I_{\mathcal{H}_2} \end{pmatrix} = \begin{pmatrix} A_{11} & \frac{1}{\mu} A_{12} \\ \mu A_{21} & A_{22} \end{pmatrix}$$

are $\mu\delta_{21}$ in the first column and $(1/\mu)\delta_{12}$ in the second. By **[13,** Theorem 2.2.8], $\mathcal{A}_{22}(\mu)$ is closable and closed if A_{11}, A_{22} are closed, provided that $(\mu\delta_{21})^2(1 + (\delta_{12}/\mu)^2) < 1$, or equivalently,

$$\delta_{12}^2\delta_{21}^2 + \mu^2\delta_{21}^2 < 1. \tag{3.6}$$

Since $\delta_{12}\delta_{21} < 1$ by (3.4) for $k = 2$, we have $\mu_2 := (1 - \delta_{12}^2\delta_{21}^2)/\delta_{21}^2 > 0$ and (3.6) holds for all $\mu \in (0, \mu_2)$.

Now let $n \geq 3$, and assume that $(\text{i})_{n-1}$ and $(\text{ii})_{n-1}$ hold.

$(\text{i})_n$ First, we show that, if we decompose the 2×2 operator matrix $\mathcal{A}_{n-1,n-1}(\mu)$ in $\widetilde{\mathcal{H}}_{n-1} = \widetilde{\mathcal{H}}_{n-2} \oplus \mathcal{H}_{n-1}$ as $\mathcal{A}_{n-1,n-1}(\mu) = \mathcal{A}^0_{n-1,n-1}(\mu) + \mathcal{A}^1_{n-1,n-1}(\mu)$ with

$$\begin{aligned} \mathcal{A}^0_{n-1,n-1}(\mu) &:= \begin{pmatrix} \mathcal{A}_{n-2,n-2}(\mu) & \mathcal{A}_{n-2,n-1}(\mu) \\ 0 & A_{n-1,n-1} \end{pmatrix}, \\ \mathcal{A}^1_{n-1,n-1}(\mu) &:= \begin{pmatrix} 0 & 0 \\ \mathcal{A}_{n-1,n-2}(\mu) & 0 \end{pmatrix}, \end{aligned} \tag{3.7}$$

then $\mathcal{A}^1_{n-1,n-1}(\mu)$ is $\mathcal{A}^0_{n-1,n-1}(\mu)$-bounded with some $\mathcal{A}^0_{n-1,n-1}(\mu)$-bound $\delta^{1,0}_{n-1}(\mu) < 1$. Toward this end, let $\varepsilon > 0$ be arbitrary. By induction hypothesis $(\text{i})_{n-1}$, there is a $\mu_{n-1} > 0$ such that the operator $\mathcal{A}_{n-1,n-2}(\mu)$ is $\mathcal{A}_{n-2,n-2}(\mu)$-bounded with $\mathcal{A}_{n-2,n-2}(\mu)$-bound $\widehat{\delta}_{n-1}(\mu)$ for every $\mu \in (0, \mu_{n-1})$. By assumption, the operator $A_{n-2,n-1}$ is $A_{n-1,n-1}$-bounded with $A_{n-1,n-1}$-bound $\delta_{n-2,n-1}$, and

$\|\mathcal{A}_{n-2,n-1}(\mu)f_{n-1}\| = \|(1/\mu)A_{n-2,n-1}f_{n-1}\|$ by definition. Thus, there are constants $\widehat{a}'_{n-1}(\mu)$, $\widehat{b}'_{n-1}(\mu)$, $a'_{n-2,n-1}$, $b'_{n-2,n-1} \geq 0$ with

$$\begin{aligned}\widehat{\delta}_{n-1}(\mu) \leq \widehat{b}'_{n-1}(\mu) < \widehat{\delta}_{n-1}(\mu) + \varepsilon,\\ \delta_{n-2,n-1} \leq b'_{n-2,n-1} < \delta_{n-2,n-1} + \varepsilon\end{aligned}$$

such that, for $\widetilde{f}_{n-1} = (\widetilde{f}_{n-2}\, f_{n-1})^{\mathrm{t}} \in D(\mathcal{A}^0_{n-1,n-1}(\mu)) \subset \widehat{\mathcal{H}}_{n-1} = \widetilde{\mathcal{H}}_{n-2} \oplus \mathcal{H}_{n-1}$,

$$\begin{aligned}
&\|\mathcal{A}^1_{n-1,n-1}(\mu)\widetilde{f}_{n-1}\|^2 = \|\mathcal{A}_{n-1,n-2}(\mu)\widetilde{f}_{n-2}\|^2\\
&\quad \leq \widehat{a}'^2_{n-1}(\mu)\|\widetilde{f}_{n-2}\|^2 + \widehat{b}'^2_{n-1}(\mu)\|\mathcal{A}_{n-2,n-2}(\mu)\widetilde{f}_{n-2}\|^2\\
&\quad \leq \widehat{a}'^2_{n-1}(\mu)\|\widetilde{f}_{n-2}\|^2 + \widehat{b}'_{n-1}(\mu)^2\\
&\qquad \cdot \left(\|\mathcal{A}_{n-2,n-2}(\mu)\widetilde{f}_{n-2} + \mathcal{A}_{n-2,n-1}(\mu)f_{n-1}\| + \left\|\frac{1}{\mu}A_{n-2,n-1}f_{n-1}\right\|\right)^2\\
&\quad \leq \widehat{a}'^2_{n-1}(\mu)\|\widetilde{f}_{n-2}\|^2 + \widehat{b}'^2_{n-1}(\mu)\\
&\qquad \cdot \left(\left(1+\frac{1}{\gamma}\right)\|\mathcal{A}_{n-2,n-2}(\mu)\widetilde{f}_{n-2}+\mathcal{A}_{n-2,n-1}(\mu)f_{n-1}\|^2\right.\\
&\qquad\quad \left. + (1+\gamma)\frac{1}{\mu^2}(a'^2_{n-2,n-1}\|f_{n-1}\|^2 + b'^2_{n-2,n-1}\|A_{n-1,n-1}f_{n-1}\|^2)\right)\\
&\quad \leq \max\left\{\widehat{a}'^2_{n-1}(\mu), \widehat{b}'^2_{n-1}(\mu)(1+\gamma)\frac{1}{\mu^2}a'^2_{n-2,n-1}\right\}\|\widetilde{f}_{n-1}\|^2\\
&\qquad + \widehat{b}'^2_{n-1}(\mu)\max\left\{\left(1+\frac{1}{\gamma}\right), (1+\gamma)\frac{1}{\mu^2}b'^2_{n-2,n-1}\right\}\|\mathcal{A}^0_{n-1,n-1}(\mu)\widetilde{f}_{n-1}\|^2
\end{aligned}$$

with arbitrary $\gamma > 0$. The second maximum becomes minimal if we choose $\gamma^{-1} := (1/\mu^2)b'^2_{n-2,n-1}$. Thus, $\mathcal{A}^1_{n-1,n-1}(\mu)$ is $\mathcal{A}^0_{n-1,n-1}(\mu)$-bounded with $\mathcal{A}^0_{n-1,n-1}(\mu)$-bound

$$\begin{aligned}\delta^{1,0}_{n-1}(\mu) &\leq \widehat{b}'_{n-1}(\mu)\sqrt{1 + \frac{1}{\mu^2}b'^2_{n-2,n-1}}\\ &\leq (\widehat{\delta}_{n-1}(\mu) + \varepsilon)\sqrt{1 + \frac{1}{\mu^2}(\delta_{n-2,n-1} + \varepsilon)^2}.\end{aligned}$$

Since $\varepsilon>0$ is arbitrary and due to estimate (3.5) from $(\mathrm{i})_{n-1}$, we find

$$
\begin{aligned}
\delta^{1,0}_{n-1}(\mu) &\le \widehat{\delta}_{n-1}(\mu)\sqrt{1+\frac{1}{\mu^2}\delta^2_{n-2,n-1}}\\
&\le \delta_{n-1,n-2}\sqrt{\mu^2+\delta^2_{n-2,n-1}}\,\frac{1}{1-\delta_{n-2}(\mathcal{A},\mu)}\\
&= \delta_{n-1}(\mathcal{A},\mu).
\end{aligned}
\tag{3.8}
$$

By assumption (3.4) for $k=n-1$, we have $\delta_{n-1}(\mathcal{A},0)=\delta_{n-1}(\mathcal{A})<1$. Hence, since $\delta_{n-1}(\mathcal{A},\cdot)$ is continuous in a neighborhood of 0, we can choose $\widetilde{\mu}_n>0$ so that $\delta^{1,0}_{n-1}(\mu)\le\delta_{n-1}(\mathcal{A},\mu)<1$ for all $\mu\in(0,\widetilde{\mu}_n)$.

Thus, if we set $\widehat{\mu}_n:=\min\{\mu_{n-1},\widetilde{\mu}_n\}$, then the operator $\mathcal{A}^1_{n-1,n-1}(\mu)$ is $\mathcal{A}_{n-1,n-1}(\mu)$-bounded for $\mu\in(0,\widehat{\mu}_n)$ with $\mathcal{A}_{n-1,n-1}(\mu)$-bound

$$
\delta^1_{n-1}(\mu)\le\frac{\delta^{1,0}_{n-1}(\mu)}{1-\delta^{1,0}_{n-1}(\mu)}\le\frac{\delta_{n-1}(\mathcal{A},\mu)}{1-\delta_{n-1}(\mathcal{A},\mu)};
\tag{3.9}
$$

here, we have used [**13,** Lemma 2.1.6] and (3.8).

By definition and assumption, $\mathcal{A}_{n,n-1}(\mu)$ is A_{nn}-bounded with A_{nn}-bound $\mu\delta_{n,n-1}$. By what was shown above, $\mathcal{A}^1_{n-1,n-1}(\mu)$ is $\mathcal{A}_{n-1,n-1}(\mu)$-bounded for $\mu\in(0,\widehat{\mu}_n)$ with $\mathcal{A}_{n-1,n-1}(\mu)$-bound $\delta^1_{n-1}(\mu)$ satisfying estimate (3.9); note that, by definition (3.7), we have $\|\mathcal{A}^1_{n-1,n-1}(\mu)\widetilde{f}_{n-1}\|=\|\mathcal{A}_{n-1,n-2}(\mu)\widetilde{f}_{n-2}\|$ for $\widetilde{f}_{n-1}=(\widetilde{f}_{n-2}\ f_{n-1})^{\mathrm{t}}\in D(\mathcal{A}_{n-1,n-2}(\mu))\oplus\mathcal{H}_{n-1}$. Hence, there exist constants $a'_{n,n-1}$, $b'_{n,n-1}$, $a^1_{n-1}(\mu)$, $b^1_{n-1}(\mu)\ge0$ with

$$
\delta_{n,n-1}\le b'_{n,n-1}<\delta_{n,n-1}+\varepsilon,\qquad \delta^1_{n-1}(\mu)\le b^1_{n-1}(\mu)<\delta^1_{n-1}(\mu)+\varepsilon
$$

so that, for $\widetilde{f}_{n-1}=(f_i)_{i=1}^{n-1}=(\widetilde{f}_{n-2}\ f_{n-1})^{\mathrm{t}}\in D(\mathcal{A}_{n-1,n-1})\subset\widetilde{\mathcal{H}}_{n-1}=\widetilde{\mathcal{H}}_{n-2}\oplus\mathcal{H}_{n-1}$,

$$
\begin{aligned}
&\|\mathcal{A}_{n,n-1}(\mu)\widetilde{f}_{n-1}\|^2=\|\mu A_{n,n-1}f_{n-1}\|^2\\
&\quad\le\mu^2\big(a'^2_{n,n-1}\|f_{n-1}\|^2+b'^2_{n,n-1}\|A_{n-1,n-1}f_{n-1}\|^2\big)\\
&\quad\le\mu^2(a'^2_{n,n-1}\|f_{n-1}\|^2+b'^2_{n,n-1}\\
&\qquad\cdot(\|\mathcal{A}_{n-1,n-2}(\mu)\widetilde{f}_{n-2}+A_{n-1,n-1}f_{n-1}\|+\|\mathcal{A}^1_{n-1,n-1}(\mu)\widetilde{f}_{n-1}\|)^2)\\
&\quad\le\mu^2(a'^2_{n,n-1}\|f_{n-1}\|^2+b'^2_{n,n-1})\\
&\qquad\cdot\Big(\Big(1+\frac{1}{\gamma}\Big)\|\mathcal{A}_{n-1,n-2}(\mu)\widetilde{f}_{n-2}+A_{n-1,n-1}f_{n-1}\|^2
\end{aligned}
$$

$$
\begin{aligned}
&\quad + (1+\gamma)(a^1_{n-1}(\mu)^2\|\widetilde{f}_{n-1}\|^2 + b^1_{n-1}(\mu)^2\|\mathcal{A}_{n-1,n-1}(\mu)\widetilde{f}_{n-1}\|^2)\Big)\\
&\le \mu^2(a'^2_{n,n-1} + b'^2_{n,n-1}(1+\gamma)a^1_{n-1}(\mu)^2)\|\widetilde{f}_{n-1}\|^2\\
&\quad + \mu^2\, b'^2_{n,n-1}\Big(\Big(1+\frac{1}{\gamma}\Big) + (1+\gamma)b^1_{n-1}(\mu)^2\Big)\|\mathcal{A}_{n-1,n-1}(\mu)\widetilde{f}_{n-1}\|^2
\end{aligned}
$$

with arbitrary $\gamma > 0$; here, in the last step, the obvious inequality $\|\mathcal{A}_{n-1,n-2}(\mu)\widetilde{f}_{n-2} + A_{n-1,n-1}f_{n-1}\|^2 \le \|\mathcal{A}_{n-1,n-1}(\mu)\widetilde{f}_{n-1}\|^2$ was used. The last factor but one becomes minimal for $\gamma^{-1} := b^1_{n-1}(\mu)^2$. In the same manner as above, since $\varepsilon > 0$ is arbitrary, we see that $\mathcal{A}_{n,n-1}(\mu)$ is $\mathcal{A}_{n-1,n-1}(\mu)$-bounded for $\mu \in (0, \widehat{0}\mu_n)$ with $\mathcal{A}_{n-1,n-1}(\mu)$-bound

$$
\widehat{\delta}_n(\mu) \le \mu\delta_{n,n-1}\big(1+\delta^1_{n-1}(\mu)\big) \le \mu\delta_{n,n-1}\frac{1}{1-\delta_{n-1}(\mathcal{A},\mu)},
$$

where we have used estimate (3.9) for $\delta^1_{n-1}(\mu)$. This completes the proof of $(\mathrm{i})_n$.

$(\mathrm{ii})_n$ We apply the claim for $n = 2$ to the operator matrix

$$
\mathcal{A}_{nn}(\mu) = \begin{pmatrix} \mathcal{A}_{n-1,n-1}(\mu) & \mathcal{A}_{n-1,n}(\mu) \\ \mathcal{A}_{n,n-1}(\mu) & A_{nn} \end{pmatrix}.
$$

By induction hypothesis $(\mathrm{ii})_{n-1}$, there exists a $\mu_{n-1} > 0$ such that, for $\mu \in (0, \mu_{n-1})$, the operator $\mathcal{A}_{n-1,n-1}(\mu)$ is closable, and closed if $A_{11}, \dots, A_{n-1,n-1}$ are closed, while all other entries of $\mathcal{A}_{nn}(\mu)$ are closable by assumption. Moreover, by definition and assumption, the A_{nn}-bound of $\mathcal{A}_{n-1,n}(\mu) = (1/\mu)A_{n-1,n}$ is $(1/\mu)\delta_{n-1,n}$; by $(\mathrm{i})_n$, the $\mathcal{A}_{n-1,n-1}(\mu)$-bound of $\mathcal{A}_{n,n-1}(\mu)$ is $\widehat{\delta}_n(\mu)$ for $\mu \in (0, \widehat{\mu}_n)$. Hence, by $(\mathrm{ii})_2$, there exists a $\mu_2 > 0$ such that $\mathcal{A}_{nn}(\mu)$ is closable, and closed if $A_{11}, \dots, A_{n-1,n-1}, A_{nn}$ are closed, for $\mu \in (0, \min\{\mu_2, \mu_{n-1}, \widehat{\mu}_n\})$ provided that $(1/\mu)\delta_{n-1,n}\widehat{\delta}_n(\mu) < 1$. Due to estimate (3.5) in $(\mathrm{i})_n$ already proved above, the latter holds if

$$
\widetilde{\delta}_n(\mu) := \delta_{n,n-1}\delta_{n-1,n}\frac{1}{1-\delta_{n-1}(\mathcal{A},\mu)} < 1.
$$

By condition (3.3) for $k = n$, we have $\widetilde{\delta}_n(0) = \delta_n(\mathcal{A}) < 1$. Since $\widetilde{\delta}_n(\cdot)$ is continuous in a neighborhood of 0, there exists a $\mu_n \in (0, \min\{\mu_2, \mu_{n-1}, \widehat{\mu}_n\})$ with $\widetilde{\delta}_n(\mu) < 1$ for $\mu \in (0, \mu_n)$. Hence, $\mathcal{A}_{nn}(\mu)$ is closable, and closed if $A_{11}, \dots, A_{nn}$ are closed, for $\mu \in (0, \mu_n)$. $\square$

The next corollary contains the particular cases $n = 2$ and $n = 3$ of Theorem 3.2; note that, for $n = 2$, it improves the closability/closedness criterion in [**13,** Theorem 2.2.8], which requires either $\delta_{21}\sqrt{1+\delta_{12}^2} < 1$ or $\delta_{12}\sqrt{1+\delta_{21}^2} < 1$.

Corollary 3.4. *Let $n = 2$. If*

$$\delta_{21}\delta_{12} < 1,$$

then $\mathcal{A} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$ is closable and closed if A_{ii}, $i = 1, 2$, are closed.

Let $n = 3$. If

$$\delta_{21}\delta_{12} + \delta_{32}\delta_{23} < 1, \tag{3.10}$$

then $\mathcal{A} = \begin{pmatrix} A_{11} & A_{12} & 0 \\ A_{21} & A_{22} & A_{23} \\ 0 & A_{32} & A_{33} \end{pmatrix}$ is closable and closed if A_{ii}, $i = 1, 2, 3$, are closed.

Corollary 3.5. *Let $\mathcal{A} = (A_{ij})_{i,j=1}^n$ be tridiagonal and diagonally dominant such that, for every $k = 2, 3, \dots, n$, either $A_{k,k-1}$ has $A_{k-1,k-1}$-bound 0 or $A_{k-1,k}$ has A_{kk}-bound 0, i.e.,*

$$\delta_{k,k-1} = 0 \quad \text{or} \quad \delta_{k-1,k} = 0, \qquad k = 2, 3, \dots, n. \tag{3.11}$$

Then $\mathcal{A}$ is closable and closed if its diagonal elements are closed.

Sufficient conditions for relative bound 0 include boundedness, relative compactness, and domain inclusions for some fractional power (see e.g., [**6,** Corollary III.7.7], [**13,** Corollary 2.1.20]). Therefore, the next corollary is immediate from Corollary 3.5.

Corollary 3.6. *Let $\mathcal{A} = (A_{ij})_{i,j=1}^n$ be tridiagonal and diagonally dominant such that, for every $k = 2, 3, \dots, n$, either one of*

(i) *$A_{k,k-1}$ is bounded*;
(ii) *$A_{k,k-1}$ is $A_{k-1,k-1}$-compact and $\mathcal{H}_{k-1}$, $\mathcal{H}_k$ are reflexive*;
(iii) *$\mathcal{H}_{k-1}$, $\mathcal{H}_k$ are Hilbert spaces and $D(|A_{k-1,k-1}|^\gamma) \subset D(A_{k,k-1})$ for some $\gamma \in (0, 1)$*;

or one of

(i′) *$A_{k-1,k}$ is bounded*;

(ii′) *$A_{k-1,k}$ is $A_{k,k}$-compact and $\mathcal{H}_{k-1}$, $\mathcal{H}_k$ are reflexive;*
(iii′) *$\mathcal{H}_{k-1}$, $\mathcal{H}_k$ are Hilbert spaces and $D(|A_{k,k}|^\gamma) \subset D(A_{k-1,k})$ for some $\gamma \in (0,1)$;*

holds. Then, $\mathcal{A}$ is closable and closed if its diagonal elements are closed.

Remark 3.7. Theorem 3.2 and its corollaries also apply to $n \times n$ operator matrices that are similar to diagonally dominant tridiagonal ones; in particular, they apply to operator matrices $\mathcal{A}$ in $\mathcal{H} = \mathcal{H}_1 \oplus \cdots \oplus \mathcal{H}_n$ for which there exists a permutation $(i_1, i_2, \ldots, i_n)$ of $(1, 2, \ldots, n)$ such that the matrix representation of $\mathcal{A} = (A_{i_k i_l})_{k,l=1}^n$ with respect to $\mathcal{H} = \mathcal{H}_{i_1} \oplus \cdots \oplus \mathcal{H}_{i_n}$ is tridiagonal and diagonally dominant.

4. The block numerical range. The block numerical range for 2×2 operator matrices, also called the quadratic numerical range, was introduced in [**11**] and further studied in a series of papers, in particular, in [**10, 14**] (also see [**13**]); in the latter two, the spectral inclusion property was proved for diagonally dominant and off-diagonally dominant operator matrices. The block numerical range for arbitrary n was introduced in [**15**] for bounded entries.

In this section, we generalize the block numerical range to $n \times n$ operator matrices with unbounded entries and study some of its elementary properties. From now on, we assume that $\mathcal{H}_1, \ldots, \mathcal{H}_n$ are Hilbert spaces; by $(\cdot,\cdot)_i$ and $\|\cdot\|_i$ we denote the scalar product and corresponding norm in $\mathcal{H}_i$, $i = 1, \ldots, n$, respectively.

Definition 4.1. Let $\mathcal{A} = (A_{ij})_{i,j=1}^n$ be an operator matrix in the Hilbert space $\mathcal{H} = \mathcal{H}_1 \oplus \cdots \oplus \mathcal{H}_n$ with domain

$$D(\mathcal{A}) = \bigoplus_{j=1}^n \left(\bigcap_{i=1}^n D(A_{ij}) \right).$$

Set

$$\begin{aligned} \mathbb{S}^n &:= \mathbb{S}_{\mathcal{H}_1 \oplus \cdots \oplus \mathcal{H}_n} \\ &:= \{ f = (f_1, \ldots, f_n)^t \in \mathcal{H}_1 \oplus \cdots \oplus \mathcal{H}_n,\ \|f_j\|_j = 1,\ j = 1, \ldots, n \}. \end{aligned}$$

For $f \in D(\mathcal{A}) \cap \mathbb{S}^n$, we define the $n \times n$ matrix

$$\mathcal{A}_f := \big((A_{ij} f_j, f_i)_i \big)_{i,j=1}^n \in M_n(\mathbb{C}).$$

Then, we call the set of eigenvalues of all of these matrices,

$$W_{\mathcal{H}_1 \oplus \cdots \oplus \mathcal{H}_n}(\mathcal{A}) := \bigcup \big\{ \sigma_{\mathrm{p}}(\mathcal{A}_f) : f \in D(\mathcal{A}) \cap \mathbb{S}^n \big\},$$

the *block numerical range* of $\mathcal{A}$ with respect to $\mathcal{H} = \mathcal{H}_1 \oplus \cdots \oplus \mathcal{H}_n$; for a fixed decomposition of $\mathcal{H}$, we also write

$$W^n(\mathcal{A}) = W_{\mathcal{H}_1 \oplus \cdots \oplus \mathcal{H}_n}(\mathcal{A}).$$

Note that in the definition of $W^n(\mathcal{A})$ it is not only required that $f \neq 0$, but even $f_j \neq 0$, $j = 1, \ldots, n$. Occasionally the following, clearly equivalent, description of the block numerical range is useful.

Remark 4.2. For $f = (f_j)_{j=1}^n \in D(\mathcal{A})$, $f_j \neq 0$, $j = 1, \ldots, n$, we set

$$\mathcal{A}_f := \left(\frac{(A_{ij} f_j, f_i)_i}{\|f_i\|_i \|f_j\|_j} \right)_{i,j=1}^n \in M_n(\mathbb{C})$$

and

$$\Delta(f_1, \ldots, f_n; \lambda) := \|f_1\|_1^2 \cdots \|f_n\|_n^2 \, \det(\mathcal{A}_f - \lambda I_{\mathbb{C}^n}), \quad \lambda \in \mathbb{C}.$$

Then,

$$\begin{aligned} W^n(\mathcal{A}) &= \bigcup \big\{ \sigma_{\mathrm{p}}(\mathcal{A}_f) : f = (f_j)_{j=1}^n \in D(\mathcal{A}), f_j \neq 0, \, j = 1, \ldots, n \big\} \\ &= \big\{ \lambda \in \mathbb{C} : \text{there exists an } f = (f_j)_{j=1}^n \in D(\mathcal{A}), \\ &\qquad\qquad f_j \neq 0, \, j = 1, \ldots, n, \ \det(\mathcal{A}_f - \lambda I_{\mathbb{C}^n}) = 0 \big\} \\ &= \big\{ \lambda \in \mathbb{C} : \text{there exists an } f = (f_j)_{j=1}^n \in D(\mathcal{A}), \\ &\qquad\qquad f_j \neq 0, \, j = 1, \ldots, n, \ \Delta(f_1, \ldots, f_n; \lambda) = 0 \big\}. \end{aligned}$$

Remark 4.3.

(i) For $n = 1$, the block numerical range coincides with the usual numerical range of $\mathcal{A}$, given by

$$W(\mathcal{A}) := \{ (\mathcal{A} f, f) : f \in D(\mathcal{A}), \, \|f\| = 1 \};$$

for $n = 2$ it is the quadratic numerical range introduced in **[11]**.

(ii) If $\mathcal{A}$ is a lower or upper tridiagonal matrix, then

$$W^n(\mathcal{A}) = W(A_{11}) \cup \cdots \cup W(A_{nn}).$$

(iii) If $\mathcal{A}$ is symmetric, $\mathcal{A} \subset \mathcal{A}^*$, then $W^n(\mathcal{A}) \subset \mathbb{R}$.

The next proposition is a straightforward generalization of the fact that the numerical range contains the quadratic numerical range (see [**13,** Proposition 3.2]).

Proposition 4.4. *$W^n(\mathcal{A}) \subset W(\mathcal{A})$.*

Proof. If $\lambda_0 \in W^n(\mathcal{A})$, there exist $f = (f_j)_{j=1}^n \in D(\mathcal{A}) \cap \mathbb{S}^n$ and $c = (c_j)_{j=1}^n \in \mathbb{C}^n$ with $\|c\| := \sqrt{|c_1|^2 + \cdots + |c_n|^2} = 1$ such that $\mathcal{A}_f c = \lambda_0 c$. Then, the vector $f_c := (c_j f_j)_{j=1}^n \in D(\mathcal{A})$ satisfies

$$\|f_c\|^2 = \sum_{j=1}^n \|c_j f_j\|_j^2 = \sum_{j=1}^n |c_j|^2 = 1,$$

and we have $\lambda_0 = (\mathcal{A}_f c, c) = (\mathcal{A} f_c, f_c) \in W(\mathcal{A})$. $\square$

The next proposition shows that, for a diagonally dominant $n \times n$ operator matrix $\mathcal{A}$, the block numerical range of a principal minor is contained in $W^n(\mathcal{A})$ if a certain dimension condition holds (see [**15,** Theorem 3.1]); for non-diagonally dominant matrices, this inclusion only holds for the closures.

Proposition 4.5. *Let $k \in \mathbb{N}$, $1 \le k \le n$, and $\mathcal{I} := \{i_1, \ldots, i_k\} \subset \mathbb{N}$, $1 \le i_1 < \cdots < i_k \le n$. Denote by $P_{\mathcal{I}} : \mathcal{H}_1 \oplus \cdots \oplus \mathcal{H}_n \longrightarrow \mathcal{H}_{i_1} \oplus \cdots \oplus \mathcal{H}_{i_k}$ the projection onto the components $i_1, \ldots, i_k$ of $\mathcal{H}$. If $\bigoplus_{i \in \mathcal{I}} D(A_{ii})$ is a core for $P_{\mathcal{I}} \mathcal{A} P_{\mathcal{I}}^*$ and there exists an enumeration $i'_1, \ldots, i'_{n-k}$ of the elements of the set $\{1, \ldots, n\} \setminus \{i_1, \ldots, i_k\} =: \{i'_1, \ldots, i'_{n-k}\}$, with $\dim \mathcal{H}_{i'_j} > n - j$, $j = 1, \ldots, n-k$, then*

$$\overline{W_{\mathcal{H}_{i_1} \oplus \cdots \oplus \mathcal{H}_{i_k}}(P_{\mathcal{I}} \mathcal{A} P_{\mathcal{I}}^*)} \subset \overline{W_{\mathcal{H}_1 \oplus \cdots \oplus \mathcal{H}_n}(\mathcal{A})};$$

if $\mathcal{A}$ is diagonally dominant, then

$$W_{\mathcal{H}_{i_1} \oplus \cdots \oplus \mathcal{H}_{i_k}}(P_{\mathcal{I}} \mathcal{A} P_{\mathcal{I}}^*) \subset W_{\mathcal{H}_1 \oplus \cdots \oplus \mathcal{H}_n}(\mathcal{A}).$$

Proof. The domain of $\mathcal{A}_{\mathcal{I}} := P_{\mathcal{I}} \mathcal{A} P_{\mathcal{I}}^*$ satisfies

$$D(\mathcal{A}_{\mathcal{I}}) = \bigoplus_{j \in \mathcal{I}} \bigcap_{i \in I} D(A_{ij}) \supset \bigoplus_{j \in \mathcal{I}} \bigcap_{i=1}^n D(A_{ij}) = \bigoplus_{j \in \mathcal{I}} D_j.$$

Both claims follow if we show that

$$W_{\mathcal{H}_{i_1}\oplus\cdots\oplus\mathcal{H}_{i_k}}(P_{\mathcal{I}}\mathcal{A}P_{\mathcal{I}}^*|_{\oplus_{i\in\mathcal{I}}D_i}) \subset W_{\mathcal{H}_1\oplus\cdots\oplus\mathcal{H}_n}(\mathcal{A}). \tag{4.1}$$

Indeed, the core property of $\bigoplus_{i\in\mathcal{I}} D_i$ yields that

$$\begin{aligned}\overline{W_{\mathcal{H}_{i_1}\oplus\cdots\oplus\mathcal{H}_{i_k}}(P_{\mathcal{I}}\mathcal{A}P_{\mathcal{I}}^*)} &= \overline{W_{\mathcal{H}_{i_1}\oplus\cdots\oplus\mathcal{H}_{i_k}}(P_{\mathcal{I}}\mathcal{A}P_{\mathcal{I}}^*|_{\oplus_{i\in\mathcal{I}}D_i})}\\ &\subset \overline{W_{\mathcal{H}_1\oplus\cdots\oplus\mathcal{H}_n}(\mathcal{A})};\end{aligned}$$

if $\mathcal{A}$ is diagonally dominant, then

$$D(\mathcal{A}_{\mathcal{I}}) = \bigoplus_{i\in\mathcal{I}} D(A_{ii}) = \bigoplus_{i\in\mathcal{I}} D_i,$$

and therefore, $P_{\mathcal{I}}\mathcal{A}P_{\mathcal{I}}^*|_{\oplus_{i\in\mathcal{I}}D_i} = P_{\mathcal{I}}\mathcal{A}P_{\mathcal{I}}^*$ in (4.1).

We prove (4.1) inductively. For $k = n$, the claim is trivial. For $k = n-1$ there is an $l \in \{1,\dots,n\}$ such that $\mathcal{I}\cup\{\ell\} = \{i_1,\dots,i_{n-1}\}\cup\{\ell\} = \{1,\dots,n\}$. Then, $\mathcal{A}_{\mathcal{I}} = P_{\mathcal{I}}\mathcal{A}P_{\mathcal{I}}^*$ arises from $\mathcal{A}$ by deleting the ℓth row and column. Denote $\mathcal{H}_I := \mathcal{H}_1\oplus\cdots\oplus\mathcal{H}_{\ell-1}\oplus\mathcal{H}_{\ell+1}\oplus\cdots\oplus\mathcal{H}_n$. If $\lambda \in W_{\mathcal{H}_1\oplus\cdots\oplus\mathcal{H}_{\ell-1}\oplus\mathcal{H}_{\ell+1}\oplus\cdots\oplus\mathcal{H}_n}(\mathcal{A}_{\mathcal{I}}|_{\oplus_{i\in\mathcal{I}}D_i})$, then there exists an element $x' = (x_1\cdots x_{\ell-1}\,x_{\ell+1}\cdots x_n)^{\mathrm{t}} \in \mathbb{S}_{\mathcal{H}_I}$, $x' \in \bigoplus_{i\in\mathcal{I}} D_i = \bigoplus_{\substack{i=1\\ i\neq\ell}}^n \bigcap_{\substack{j=1\\ j\neq\ell}}^n D(A_{ij})$, such that $\det\big((\mathcal{A}_{\mathcal{I}})_{x'}-\lambda\big) = 0$. By assumption,

$$\dim\operatorname{span}\big\{A_{\ell 1}x_1,\dots,A_{\ell,\ell-1}x_{\ell-1},A_{\ell,\ell+1}x_{\ell+1},\dots,A_{in}x_n\big\} \le n-1 < \dim\mathcal{H}_\ell$$

and $D_\ell = \bigcap_{j=1}^n D(A_{\ell j}) \subset \mathcal{H}_\ell$ is dense. Hence, there exists an $x_\ell \in D_\ell$, $\|x_\ell\|_\ell = 1$, with

$$(\mathcal{A}_x)_{\ell j} = (A_{\ell j}x_j, x_\ell)_\ell = 0, \quad j = 1,\dots,\ell-1,\ \ell+1,\dots,n.$$

Indeed, there exist an orthonormal system $\{e_1,\dots,e_n\} \subset \mathcal{H}_\ell$ and $\{e_1',\dots,e_n'\} \subset D_\ell$ with $\|e_i - e_i'\|_\ell < 1/2$, $i = 1,2,\dots,n$. Then, $\{e_1',e_2',\dots,e_n'\}$ are linearly independent. If we let $y = \alpha_1e_1'+\cdots+\alpha_ne_n' \in D_\ell$, then the conditions $(A_{\ell j}x_j,x_\ell)_\ell = 0$, $j = 1,\dots,\ell-1$, $\ell+1,\dots,n$, yield a system of $n-1$ linear equations for the n coefficients $\alpha_1,\dots,\alpha_n$. Choosing a non-trivial solution, we find that $x_\ell := y/\|y\|$ has the desired properties.

With this choice of x_ℓ, we have $x := (x_1, \dots, x_{\ell-1}, x_\ell, x_{\ell+1}, \dots, x_n)^t \in \mathbb{S}_{\mathcal{H}} \cap D(\mathcal{A})$ and

$$\mathcal{A}_x =$$
$$\begin{pmatrix} (\mathcal{A}_x)_{11} & \cdots & (\mathcal{A}_x)_{1,\ell-1} & (\mathcal{A}_x)_{1\ell} & (\mathcal{A}_x)_{1,\ell+1} & \cdots & (\mathcal{A}_x)_{1n} \\ \vdots & & \vdots & \vdots & \vdots & & \\ (\mathcal{A}_x)_{\ell-1,1} & \cdots & (\mathcal{A}_x)_{\ell-1,\ell-1} & (\mathcal{A}_x)_{\ell-1,\ell} & (\mathcal{A}_x)_{\ell-1,\ell+1} & \cdots & (\mathcal{A}_x)_{\ell-1,n} \\ 0 & \cdots & 0 & (\mathcal{A}_x)_{\ell\ell} & 0 & \cdots & 0 \\ (\mathcal{A}_x)_{\ell+1,1} & \cdots & (\mathcal{A}_x)_{\ell+1,\ell-1} & (\mathcal{A}_x)_{\ell+1,\ell} & (\mathcal{A}_x)_{\ell+1,\ell+1} & \cdots & (\mathcal{A}_x)_{\ell+1,n} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ (\mathcal{A}_x)_{n1} & \cdots & (\mathcal{A}_x)_{n,\ell-1} & (\mathcal{A}_x)_{n\ell} & (\mathcal{A}_x)_{n,\ell+1} & \cdots & (\mathcal{A}_x)_{nn} \end{pmatrix}.$$

Thus, $\det(\mathcal{A}_x - \lambda) = \big((\mathcal{A}_x)_{\ell\ell} - \lambda\big) \det\big((\mathcal{A}_{\mathcal{I}})_{x'} - \lambda\big) = 0$ and, therefore, $\lambda \in W_{\mathcal{H}_1 \oplus \cdots \oplus \mathcal{H}_n}(\mathcal{A})$. In the same manner, the case $k < n-1$ follows by induction. $\square$

Remark 4.6. Note that the assumption that $\mathcal{A}$ is diagonally dominant for the last claim of Proposition 4.5 is missing in the case $n = 2$ in [**13,** Theorem 2.5.4], [**15,** Theorem 3.1].

We mention the dimension condition in Proposition 4.5 always holds if all components $\mathcal{H}_i$, $i = 1, \dots, n$, have infinite dimension. If $\mathcal{H}_i$ is finite-dimensional for some $i \in \{1, \dots, n\}$, then these conditions are necessary (for counter-examples see [**13,** Example 1.11.11]).

5. Spectral inclusion. One of the most important features of the numerical range of a linear operator $\mathcal{A}$ is the spectral inclusion property:

$$\sigma_{\mathrm{p}}(\mathcal{A}) \subset W(\mathcal{A}), \qquad \sigma_{\mathrm{app}}(\mathcal{A}) \subset \overline{W(\mathcal{A})};$$

if each of the (at most two) components of $\mathbb{C} \setminus W(\mathcal{A})$ contains a point $\mu \in \rho(\mathcal{A})$, then

$$\sigma(\mathcal{A}) \subset \overline{W(\mathcal{A})}$$

(see, e.g., [**9,** Theorem 5.3.2]). Here, the approximate point spectrum of $\mathcal{A}$ is defined as

$$\sigma_{\mathrm{app}}(\mathcal{A}) := \big\{\lambda \in \mathbb{C} : \text{ there exist } (f^{(\nu)})_1^\infty \subset D(\mathcal{A}),\ \|f^{(\nu)}\| = 1, \\ (\mathcal{A} - \lambda) f^{(\nu)} \to 0,\ \nu \to \infty\big\}.$$

In the following, we prove analogs of these spectral inclusions for the block numerical range of diagonally dominant unbounded $n \times n$ operator matrices.

Theorem 5.1. *$\sigma_{\mathrm{p}}(\mathcal{A}) \subset W^n(\mathcal{A})$.*

Proof. The proof of Theorem 5.1 is completely analogous to that in the case when $\mathcal{A}$ is bounded (see **[15]**), the only difference being that the eigenvectors now belong to $D(\mathcal{A})$. □

Theorem 5.2. *If $\mathcal{A}$ is a diagonally dominant $n \times n$ operator matrix of order* 0, *then*

$$\sigma_{\mathrm{app}}(\mathcal{A}) \subset \overline{W^n(\mathcal{A})}.$$

If Ω is a component of $\mathbb{C}\backslash\overline{W^n(\mathcal{A})}$ that contains a point $\mu \in \rho(\mathcal{A})$, then $\Omega \subset \rho(\mathcal{A})$; in particular, if every component of $\mathbb{C} \subset \overline{W^n(\mathcal{A})}$ contains a point $\mu \in \rho(\mathcal{A})$, then

$$\sigma(\mathcal{A}) \subset \overline{W^n(\mathcal{A})}.$$

Proof. In the following, let $\mathcal{T} = \operatorname{diag}(A_{11}, \dots, A_{nn})$ and $\mathcal{S} = \mathcal{A} - \mathcal{T}$ be the diagonal and off-diagonal parts of $\mathcal{A}$ (see Definition 2.1).

Let $\lambda_0 \in \sigma_{\mathrm{app}}(\mathcal{A})$. Then, there exists a sequence

$$(f^{(\nu)})_{\nu=1}^{\infty} = ((f_1^{(\nu)}, \dots, f_n^{(\nu)})^t)_{\nu=1}^{\infty} \subset D(\mathcal{A})$$

with $\|f^{(\nu)}\| = \sqrt{\|f_1^{(\nu)}\|_1^2 + \cdots + \|f_n^{(\nu)}\|_n^2} = 1$ such that

$$(\mathcal{A} - \lambda_0)f^{(\nu)} =: h^{(\nu)} \longrightarrow 0, \quad \nu \to \infty. \tag{5.1}$$

Since $\mathcal{A}$ is diagonally dominant of order 0, the off-diagonal part $\mathcal{S}$ of $\mathcal{A}$ is $\mathcal{A}$-bounded with $\mathcal{A}$-bound 0 by Remark 2.2. Thus, (5.1) implies that $(\mathcal{S}f^{(\nu)})_1^{\infty}$ is bounded. Hence, again by (5.1), $(\mathcal{T}f^{(\nu)})_1^{\infty} = ((\mathcal{A} - \mathcal{S})f^{(\nu)})_1^{\infty}$ is also bounded, i.e., $(A_{jj}f_j^{(\nu)})_1^{\infty}$, $j = 1, \dots, n$, are bounded. Since A_{ij} is A_{jj}-bounded (see Proposition 2.3), there exist a_{ij}, $b_{ij} \geq 0$, and hence, $C > 0$, such that

$$\|A_{ij}f_j^{(\nu)}\|_i \leq a_{ij}\|f_j^{(\nu)}\|_j + b_{ij}\|A_{jj}f_j^{(\nu)}\|_j \leq C$$

for $i, j = 1, \dots, n$, $i \neq j$, $\nu \in \mathbb{N}$. In summary, we have proved that all sequences $(A_{ij}f_j^{(\nu)})_{\nu=1}^{\infty}$, $i, j = 1, \dots, n$, are bounded.

Now, we choose $\widehat{f}^{\,(\nu)} = (\widehat{f_1}^{(\nu)}, \dots, \widehat{f_n}^{(\nu)})^t \in D(\mathcal{A}) \cap \mathbb{S}^n$ such that

$$f_j^{(\nu)} = \|f_j^{(\nu)}\|_j \widehat{f_j}^{(\nu)}, \quad j = 1, \dots, n; \tag{5.2}$$

note that $\widehat{f_j}^{(\nu)}$ can be arbitrarily chosen if $f_j^{(\nu)} = 0$.

If $\liminf_{\nu\to\infty} \|f_j^{(\nu)}\|_j > 0$, $j = 1, \dots, n$, in the sequel we set $k := 0$ and $\{j_1, \dots, j_k\} := \emptyset$. If there exists a $j_1 \in \{1, \dots, n\}$ with

$$\liminf_{\nu\to\infty} \|f_{j_1}^{(\nu)}\|_{j_1} = 0,$$

then there is a subsequence $(f^{(\nu_l)})_{l=1}^\infty$ for which $\lim_{l\to\infty} \|f_{j_1}^{(\nu_l)}\|_{j_1} = 0$. If there exists a $j_2 \in \{1, \dots, n\} \setminus \{j_1\}$ with $\liminf_{l\to\infty} \|f_{j_2}^{(\nu_l)}\|_{j_2} = 0$, we choose a subsequence $(f_{j_2}^{(\nu_{l_m})})_{m=1}^\infty$ for which $\lim_{m\to\infty} \|f_{j_2}^{(\nu_{l_m})}\|_{j_2} = 0$. We continue this process until we have found a subsequence $(f^{(\nu_\mu)})_{\mu=1}^\infty$ and $\{j_1, \dots, j_k\} \subset \{1, \dots, n\}$ such that

$$\lim_{\mu\to\infty} \|f_j^{(\nu_\mu)}\|_j = 0, \quad j \in \{j_1, \dots, j_k\}, \tag{5.3}$$

$$\liminf_{\mu\to\infty} \|f_i^{(\nu_\mu)}\|_i > 0, \quad i \in \{1, \dots, n\} \setminus \{j_1, \dots, j_k\}. \tag{5.4}$$

Since $\|f^{(\nu)}\| = 1$, the set $\{i_1, \dots, i_{n-k}\} := \{1, \dots, n\} \setminus \{j_1, \dots, j_k\}$ is non-empty. Without loss of generality, we may assume that the sequence $(f^{(\nu)})_{\nu=1}^\infty$ itself has the above properties and, further,

$$\begin{aligned}
&\{j_1, \dots, j_k\} = \{1, \dots, k\},\\
&\{1, \dots, n\} \setminus \{j_1, \dots, j_k\} = \{k+1, \dots, n\},\\
&\dim \mathcal{H}_1 \geq \dots \geq \dim \mathcal{H}_k,\\
&\|f_i^{(\nu)}\|_i \geq \gamma_i > 0, \quad \nu \in \mathbb{N},\ i = k+1, \dots, n,
\end{aligned}$$

with positive constants γ_i, $i \in \{k+1, \dots, n\}$. Let $k_0 \in \{0, 1, \dots, k\}$ be such that

$$\dim \mathcal{H}_j > n - j, \quad j = 1, \dots, k_0, \tag{5.5}$$

$$\dim \mathcal{H}_{k_0+1} \leq n - ((k_0+1)), \tag{5.6}$$

and let P be the projection onto the last $n - k_0$ components of $\mathcal{H}$,

$$P : \mathcal{H}_1 \oplus \dots \oplus \mathcal{H}_n \longrightarrow \mathcal{H}_{k_0+1} \oplus \dots \oplus \mathcal{H}_n.$$

In the following, we will show that

$$\lambda_0 \in \overline{W^{n-k_0}(P\mathcal{A}P^*)}; \tag{5.7}$$

then, due to the choice of k_0 in (5.5) and Proposition 4.5, we have $\overline{W^{n-k_0}(P\mathcal{A}P^*)} \subset \overline{W^n(\mathcal{A})}$, and hence $\lambda_0 \in \overline{W^n(\mathcal{A})}$ as required.

In order to prove (5.7), we first show that for all non-diagonal elements in the first k_0 columns of $\mathcal{A} = (A_{ij})_{i,j=1}^n$

$$A_{ij} f_j^{(\nu)} \longrightarrow 0, \quad i = 1, \dots, n, \; j = 1, \dots, k_0, \; i \neq j. \tag{5.8}$$

For this, let $i \in \{1, \dots, n\}$, $j \in \{1, \dots, k_0\}$, $i \neq j$, and $\varepsilon > 0$ be arbitrary. Since the sequence $((A_{jj} - \lambda_0) f_j^{(\nu)})_{\nu=1}^\infty$ is bounded, there exists an $M > 0$ such that

$$\|(A_{jj} - \lambda_0) f_j^{(\nu)}\|_j \leq M, \quad \nu \in \mathbb{N}.$$

Since $\mathcal{A}$ is diagonally dominant of order 0, the operator A_{ij} is A_{jj}-bounded with A_{jj}-bound 0 if $i \neq j$. Hence, there exist constants a_{ij}, $b_{ij} \geq 0$ such that $b_{ij} < \varepsilon/(2M)$ and

$$\begin{aligned}\|A_{ij} f_j^{(\nu)}\|_i &\leq a_{ij} \|f_j^{(\nu)}\|_j + b_{ij} \|A_{jj} f_j^{(\nu)}\|_j \\ &\leq (a_{ij} + b_{ij}|\lambda_0|) \|f_j^{(\nu)}\|_j + b_{ij} \|(A_{jj} - \lambda_0) f_j^{(\nu)}\|_j, \quad \nu \in \mathbb{N}.\end{aligned}$$

Due to (5.3), we can choose $N \in \mathbb{N}$ with $\|f_j^{(\nu)}\|_j < \varepsilon/(2(a_{ij} + b_{ij}|\lambda_0|))$, $\nu \geq N$. It follows that $\|A_{ij} f_j^{(\nu)}\|_i < \varepsilon$ for $\nu \geq N$, which proves (5.8).

Using (5.8) for $i = k_0{+}1, \dots, n$ in the last $n{-}k_0$ components of (5.1), we conclude that, for $\nu \to \infty$,

$$(P\mathcal{A}P^* - \lambda_0 I_{n-k_0})(f_{k_0+1}^{(\nu)} \cdots f_n^{(\nu)})^t =: (g_{k_0+1}^{(\nu)} \cdots g_n^{(\nu)})^t \longrightarrow 0. \tag{5.9}$$

We set $\widehat{f_0}^{(\nu)} := (\widehat{f}_{k_0+1}^{(\nu)} \cdots \widehat{f}_n^{(\nu)})^t$ and consider the reduced determinant

$$\begin{aligned}\Delta(\widehat{f}_{k_0+1}^{(\nu)}, \dots, \widehat{f}_n^{(\nu)}; \lambda_0) &= \det((P\mathcal{A}P^*)_{\widehat{f}_0^{(\nu)}} - \lambda_0 I_{n-k_0}) \\ &= \det\Big(\big((\mathcal{A}_{ij} \widehat{f}_j^{(\nu)}, \widehat{f}_i^{(\nu)})_i\big)_{i,j=k_0+1}^n - \lambda_0 I_{n-k_0}\Big).\end{aligned} \tag{5.10}$$

The sequence of matrices $(((\mathcal{A}_{ij} \widehat{f}_j^{(\nu)}, \widehat{f}_i^{(\nu)})_i)_{i,j=k_0+1}^n - \lambda_0 I_{n-k_0})_{\nu=1}^\infty$ is bounded. In fact, all entries of the first $k - k_0$ columns are bounded since, by (5.6), the components $\mathcal{H}_{k_0+1}, \dots, \mathcal{H}_k$ are finite-dimensional, so that all operators A_{ij} with $i \in \{k_0 + 1, \dots, k\}$ are bounded, and

$\|\widehat{f}_j^{(\nu)}\|_j = 1$, $j = k_0+1, \dots, n$; all entries of the last $n-k$ columns are bounded since

$$\|A_{ij}\widehat{f}_j^{(\nu)}\|_i = \frac{1}{\|f_j^{(\nu)}\|_j}\|A_{ij}f_j^{(\nu)}\|_i \le \frac{1}{\gamma_j}\|A_{ij}f_j^{(\nu)}\|_i, \quad j = k+1, \dots, n,$$

and all sequences $(A_{ij}f_j^{(\nu)})_{\nu=1}^{\infty}$ were shown to be bounded (see above).

If we take the scalar product of the $n-k_0$ equations in (5.9) numbered $k_0+1, \dots, n$ with $\widehat{f}_{k_0+1}^{(\nu)}, \dots, \widehat{f}_n^{(\nu)}$ and solve, e.g., for $(A_{in}\widehat{f}_n^{(\nu)}, \widehat{f}_i^{(\nu)})_i$, $i = k_0+1, \dots, n$, we obtain

$$\begin{aligned}
&\big(A_{in}\widehat{f}_n^{(\nu)}, \widehat{f}_i^{(\nu)}\big)_i \\
&= \frac{1}{\|f_n^{(\nu)}\|_n}\Bigg(\big(g_i^{(\nu)}, \widehat{f}_i^{(\nu)}\big)_i - \sum_{j=k_0+1}^{n-1}\|f_j^{(\nu)}\|_j\big(A_{ij}\widehat{f}_j^{(\nu)}, \widehat{f}_i^{(\nu)}\big)_i + \|f_i^{(\nu)}\|_i\lambda_0\Bigg), \\
&\hspace{20em} i = k_0+1, \dots, n-1, \\
&\big(A_{nn}\widehat{f}_n^{(\nu)}, \widehat{f}_n^{(\nu)}\big)_n - \lambda_0 \\
&= \frac{1}{\|f_n^{(\nu)}\|_n}\Bigg(\big(g_n^{(\nu)}, \widehat{f}_n^{(\nu)}\big)_n - \sum_{j=k_0+1}^{n-1}\|f_j^{(\nu)}\|_j\big(A_{nj}\widehat{f}_j^{(\nu)}, \widehat{f}_n^{(\nu)}\big)_n\Bigg).
\end{aligned}$$

Adding the corresponding multiples of the first $n-k_0-1$ columns to the last column in the determinant in (5.10), we arrive at

$$\begin{aligned}
&\Delta(\widehat{f}_{k_0+1}^{(\nu)}, \dots, \widehat{f}_n^{(\nu)}; \lambda_0) \\
&= \det\Bigg(\Big(\big(A_{ij}\widehat{f}_j^{(\nu)}, \widehat{f}_i^{(\nu)}\big)_i\Big)_{i=k_0+1, j=k_0+1}^{n\quad\ \ k} \ \Big(\big(A_{ij}\widehat{f}_j^{(\nu)}, \widehat{f}_i^{(\nu)}\big)_i\Big)_{i=k_0+1, j=k+1}^{n\quad\ \ n} \\
&\hspace{16em} \frac{1}{\|f_n^{(\nu)}\|_n}\Big(\big(g_i^{(\nu)}, \widehat{f}_i^{(\nu)}\big)_i\Big)_{i=k_0+1}^{n}\Bigg).
\end{aligned}$$

The entries in all but the last column are bounded in ν, and the entries of the last column tend to 0 since $g_i^{(\nu)} \to 0$, $\nu \to \infty$, and $\|f_n^{(\nu)}\|_n \ge \gamma_n > 0$, $\nu \in \mathbb{N}$. Expanding the determinant with respect to the last column, we thus obtain

$$\Delta(\widehat{f}_{k_0+1}^{(\nu)}, \dots, \widehat{f}_n^{(\nu)}; \lambda_0) \longrightarrow 0, \quad \nu \to \infty.$$

As $\Delta(\widehat{f}^{(\nu)}_{k_0+1}, \dots, \widehat{f_n}^{(\nu)}; \cdot)$ is a monic polynomial, we can write

$$\Delta(\widehat{f}^{(\nu)}_{k_0+1}, \dots, \widehat{f_n}^{(\nu)}; \lambda) = \prod_{i=1}^{n-k_0} \big(\lambda - \lambda_i^{(\nu)}\big), \quad \nu \in \mathbb{N},$$

where $\lambda_i^{(\nu)} \in \mathbb{C}$, $i = 1, \dots, n-k_0$, are the (not necessarily disjoint) solutions of $\Delta(\widehat{f}^{(\nu)}_{k_0+1}, \dots, \widehat{f_n}^{(\nu)}; \lambda) = 0$, and thus, $\lambda_i^{(\nu)} \in W^{n-k_0}(P\mathcal{A}P^*)$, $i = 1, \dots, n-k_0$. Now, Hurwitz's theorem (see [**3,** Theorem 2.5]) implies that

$$\operatorname{dist}(\lambda_0, W^{n-k_0}(P\mathcal{A}P^*)) \le \min\{|\lambda_1^{(\nu)} - \lambda_0|, \dots, |\lambda_{n-k_0}^{(\nu)} - \lambda_0|\} \longrightarrow 0$$

for $\nu \to \infty$ and thus, $\lambda_0 \in \overline{W^{n-k_0}(P\mathcal{A}P^*)} \subset \overline{W^n(\mathcal{A})}$ by Proposition 4.5 due to the dimension conditions (5.5).

It remains to prove the last claim. By the closed graph theorem, the approximate point spectrum $\sigma_{\mathrm{app}}(\mathcal{A})$ is the complement of the set $r(\mathcal{A})$ of points of regular type of $\mathcal{A}$, $\sigma_{\mathrm{app}}(\mathcal{A}) = \mathbb{C} \setminus r(\mathcal{A})$, where

$$r(\mathcal{A}) := \{\lambda \in \mathbb{C} : \text{there is a } C_\lambda > 0 \ \|(\mathcal{A} - \lambda)f\| \ge C_\lambda \|f\|,\ f \in D(\mathcal{A})\}.$$

Since $\mathcal{A} - \lambda$ is injective for $\lambda \in r(\mathcal{A})$ and $\lambda \longmapsto \dim R(\mathcal{A} - \lambda)^\perp$ is constant on every component of $r(\mathcal{A})$, the second inclusion is a direct consequence of the first one. □

Since boundedness and relative compactness both imply relative boundedness with relative bound 0 in a Hilbert space (see [**6,** Corollary III.7.7]), the following corollary is immediate from Theorem 5.2.

Corollary 5.3. *If each A_{ij} is either bounded or A_{jj}-compact for all $i, j = 1, \dots, n$, $i \neq j$, then the spectral inclusions in Theorem* 5.2 *hold.*

In order to show that components of $\mathbb{C} \setminus \overline{W^n(\mathcal{A})}$ contain points of $\rho(\mathcal{A})$, stability results for bounded invertibility may be used (see, e.g., [**9,** Chapter IV, Section 1] or [**13,** Corollary 2.1.5]).

Remark 5.4. Since the numerical range is convex, the complement $\mathbb{C} \setminus \overline{W(\mathcal{A})}$ has at most two components. The number of components of $\mathbb{C} \setminus \overline{W^n(\mathcal{A})}$ for $n > 1$ is still unknown.

Remark 5.5. In general, the closure $\overline{\mathcal{A}}$ of $\mathcal{A}$ need not be an operator matrix so the block numerical range of $\overline{\mathcal{A}}$ is undefined. However, if $\mathcal{A}$ satisfies the assumptions of Theorem 5.2, then, due to the relations

$$\sigma_{\mathrm{p}}(\overline{\mathcal{A}}) \subset \sigma_{\mathrm{app}}(\mathcal{A}), \qquad \sigma_{\mathrm{app}}(\overline{\mathcal{A}}) = \sigma_{\mathrm{app}}(\mathcal{A}),$$

it follows that

$$\sigma_{\mathrm{p}}(\overline{\mathcal{A}}) \subset \overline{W^n(\mathcal{A})}, \qquad \sigma_{\mathrm{app}}(\overline{\mathcal{A}}) \subset \overline{W^n(\mathcal{A})}.$$

6. Inclusions among block numerical ranges and Gershgorin theorems. In this section, we first compare the spectral enclosure provided by the block numerical range with those provided by lower order block numerical ranges and, in particular, by the numerical range.

Second, we prove a row sum as well as a column sum Gershgorin theorem for diagonally dominant operator matrices. For bounded off-diagonal entries, using either this enclosure or an estimate of the block numerical range by the matrix Gershgorin theorem, we obtain an enclosure for the spectrum that depends only upon the numerical ranges of the diagonal elements and the norms of the off-diagonal entries.

If $\mathcal{H} = \mathcal{H}_1 \oplus \cdots \oplus \mathcal{H}_n$ and $\mathcal{H} = \widetilde{\mathcal{H}}_1 \oplus \cdots \oplus \widetilde{\mathcal{H}}_k$ with $n > k$, then the block numerical range $W^n(\mathcal{A})$ of an operator matrix $\mathcal{A}$ with respect to $\mathcal{H}_1 \oplus \cdots \oplus \mathcal{H}_n$ need not be contained in $W^k(\mathcal{A})$ with respect to $\widetilde{\mathcal{H}}_1 \oplus \cdots \oplus \widetilde{\mathcal{H}}_k$.

Inclusion does hold if $n \geq k$ and $\mathcal{H}_1 \oplus \cdots \oplus \mathcal{H}_n$ is a refinement of $\widetilde{\mathcal{H}}_1 \oplus \cdots \oplus \widetilde{\mathcal{H}}_k$. Thus, in this case, $W^n(\mathcal{A})$ may give a tighter spectral inclusion than $W^k(\mathcal{A})$.

Theorem 6.1. *Let $n, k \in \mathbb{N}$, $n \geq k$, and let $\mathcal{H} = \mathcal{H}_1 \oplus \cdots \oplus \mathcal{H}_n$ be a refinement of $\mathcal{H} = \widetilde{\mathcal{H}}_1 \oplus \cdots \oplus \widetilde{\mathcal{H}}_k$, that is, there exist integers $0 = i_0 < \cdots < i_k = n$ with $\widetilde{\mathcal{H}}_l = \mathcal{H}_{i_{l-1}+1} \oplus \cdots \oplus \mathcal{H}_{i_l}$, $l = 1, \ldots, k$. Then,*

$$W_{\mathcal{H}_1 \oplus \cdots \oplus \mathcal{H}_n}(\mathcal{A}) \subset W_{\widetilde{\mathcal{H}}_1 \oplus \cdots \oplus \widetilde{\mathcal{H}}_k}(\mathcal{A}),$$

or, briefly, $W^n(\mathcal{A}) \subset W^k(\mathcal{A})$ for $n \geq k$ upon refinement of the decomposition.

Proof. The proof is similar to the proof for bounded $\mathcal{A}$ (see [**13,** Theorem 1.11.13]); thus, we only sketch it. It suffices to consider the case $k = n-1$; the general case follows by induction. If $k = n-1$, there exists a $k_0 \in \{1, \dots, n-1\}$ such that

$$\begin{aligned} \widetilde{\mathcal{H}}_l &= \mathcal{H}_l, & l &\in \{1, \dots, k_0 - 1\}, \\ \widetilde{\mathcal{H}}_{k_0} &= \mathcal{H}_{k_0} \oplus \mathcal{H}_{k_0+1}, \\ \widetilde{\mathcal{H}}_l &= \mathcal{H}_{l+1}, & l &\in \{k_0 + 1, \dots, n-1\}; \end{aligned}$$

note that, for $k_0 = 1$ and $k_0 = n-1$ the first and the last set, respectively, are empty. If we apply the spectral inclusion theorem [**13,** Theorem 1.11.6] for the block numerical range of matrices to each matrix $\mathcal{A}_f$ with $f \in D(\mathcal{A}) \cap \mathbb{S}_{\mathcal{H}_1 \oplus \cdots \oplus \mathcal{H}_{k_0} \oplus \mathcal{H}_{k_0+1} \oplus \cdots \oplus \mathcal{H}_n}$, we obtain

$$\begin{aligned} W_{\mathcal{H}_1 \oplus \cdots \oplus \mathcal{H}_n}(\mathcal{A}) &= W_{\mathcal{H}_1 \oplus \cdots \oplus \mathcal{H}_{k_0} \oplus \mathcal{H}_{k_0+1} \oplus \cdots \oplus \mathcal{H}_n}(\mathcal{A}) \\ &= \bigcup \left\{ \sigma(\mathcal{A}_f) : f \in D(\mathcal{A}) \cap \mathbb{S}_{\mathcal{H}_1 \oplus \cdots \oplus \mathcal{H}_{k_0} \oplus \mathcal{H}_{k_0+1} \oplus \cdots \oplus \mathcal{H}_n} \right\} \\ &\subset \bigcup \left\{ W_{\mathbb{C} \oplus \cdots \oplus \mathbb{C}^2 \oplus \cdots \oplus \mathbb{C}}(\mathcal{A}_f) : f \in D(\mathcal{A}) \cap \mathbb{S}_{\mathcal{H}_1 \oplus \cdots \oplus \mathcal{H}_{k_0} \oplus \mathcal{H}_{k_0+1} \oplus \cdots \oplus \mathcal{H}_n} \right\}. \end{aligned}$$

Now the proof is completed in the same manner as the proof of [**13,** Theorem 1.11.13] by showing that, for every pair $f \in D(\mathcal{A}) \cap \mathbb{S}_{\mathcal{H}_1 \oplus \cdots \oplus \mathcal{H}_{k_0} \oplus \mathcal{H}_{k_0+1} \oplus \cdots \oplus \mathcal{H}_n}$ and $x \in \mathbb{C} \oplus \cdots \oplus \mathbb{C}^2 \oplus \cdots \oplus \mathbb{C}$, there exists a $g \in D(\mathcal{A}) \cap \mathbb{S}_{\mathcal{H}_1 \oplus \cdots \oplus \mathcal{H}_n}$ such that $\sigma((\mathcal{A}_f)_x) = \sigma(\mathcal{A}_g)$, and hence, $W_{\mathbb{C} \oplus \cdots \oplus \mathbb{C}^2 \oplus \cdots \oplus \mathbb{C}}(\mathcal{A}_f) \subset W_{\mathcal{H}_1 \oplus \cdots \oplus \mathcal{H}_n}(\mathcal{A})$. □

Remark 6.2. Note that the inclusion $W^n(\mathcal{A}) \subset W(\mathcal{A})$ proved in Proposition 4.4 for every decomposition $\mathcal{H} = \mathcal{H}_1 \oplus \cdots \oplus \mathcal{H}_n$ is the special case $k = 1$ of Theorem 6.1.

The classical *row sum Gershgorin theorem* for matrices (see [**8,** Theorem 6.1.1]) was extended by Salas to bounded $n \times n$ operator matrices (see [**12**] and also [**13,** Section 1.13]); the proof generalizes Householder's proof using a Neumann series argument with respect to the operator norm induced by the $|||\cdot|||_\infty$ norm on $\mathcal{H} = \mathcal{H}_1 \oplus \cdots \oplus \mathcal{H}_n$,

$$|||f|||_\infty := \max\{\|f_1\|_1, \dots, \|f_n\|_n\}, \quad f = (f_1, \dots, f_n)^t \in \mathcal{H}.$$

Here, we prove both a *row sum Gershgorin theorem* as well as a *column sum Gershgorin theorem* for unbounded diagonally dominant $n \times n$ operator matrices.

In the row sum case, we modify Householder's proof by using a different factorization in (6.2) below in which the factor diag $(A_{11}, \dots, A_{nn})$ is on the right; this allows us to use the dominance relations within columns to obtain a more elegant result.

Theorem 6.3 (Row sum Gershgorin theorem)**.** *Let $\mathcal{A}$ be a diagonally dominant $n \times n$ operator matrix with closed diagonal elements A_{jj} and A_{jj}-bounds δ_{ij} of A_{ij} such that $\sum_{\substack{j=1\\ j\neq i}}^{n} \delta_{ij} < 1$, and let a_{ij}, $b_{ij} \geq 0$ be such that $\sum_{\substack{j=1\\ j\neq i}}^{n} b_{ij} < 1$ and*

$$\|A_{ij} f_j\|_i \leq a_{ij} \|f_j\|_j + b_{ij} \|A_{jj} f_j\|_j, \quad f_j \in D(A_{jj}) \subset D(A_{ij}), \tag{6.1}$$

for $i, j = 1, 2, \dots, n$, $i \neq j$. Then, $\sigma(\mathcal{A}) \subset \mathcal{G}_{\mathrm{row}}(\mathcal{A})$ where

$$\begin{aligned}\mathcal{G}_{\mathrm{row}}(\mathcal{A}) &:= \bigcup_{j=1}^{n} \left(\sigma(A_{jj}) \cup \left\{ \lambda \in \rho(A_{jj}) : \sum_{\substack{i=1\\ i\neq j}}^{n} \big\| A_{ij}(A_{jj} - \lambda)^{-1} \big\| \geq 1 \right\} \right) \\ &\subset \bigcup_{j=1}^{n} \left(\sigma(A_{jj}) \cup \left\{ \lambda \in \rho(A_{jj}) : \big\| (A_{jj} - \lambda)^{-1} \big\| \left(\sum_{\substack{i=1\\ i\neq j}}^{n} (a_{ij} + |\lambda| b_{ij}) \right) \right.\right. \\ &\qquad\qquad \left.\left. \geq 1 - \sum_{\substack{i=1\\ i\neq j}}^{n} b_{ij} \right\} \right).\end{aligned}$$

Proof. First, we note that the assumptions on δ_{ij} imply that $\mathcal{A}$ is diagonally dominant of order < 1 and hence closed by Theorem 3.1. Suppose that $\lambda \notin \bigcup_{j=1}^{n} \sigma(A_{jj})$. Then, since $\mathcal{A}$ is diagonally dominant,

$$\mathcal{A} - \lambda = \big(I_{\mathcal{H}} + \mathcal{C}(\lambda) \big) \begin{pmatrix} A_{11} - \lambda & & 0 \\ & \ddots & \\ 0 & & A_{nn} - \lambda \end{pmatrix} \tag{6.2}$$

where $\mathcal{C}(\lambda)$ is the bounded operator matrix

$$\mathcal{C}(\lambda) := \begin{pmatrix} 0 & A_{12}(A_{22} - \lambda)^{-1} & \cdots & A_{1n}(A_{nn} - \lambda)^{-1} \\ A_{21}(A_{11} - \lambda)^{-1} & 0 & & A_{2n}(A_{nn} - \lambda)^{-1} \\ \vdots & & \ddots & \vdots \\ A_{n1}(A_{11} - \lambda)^{-1} & \cdots & \cdots & 0 \end{pmatrix}.$$

If we equip $\mathcal{H} = \mathcal{H}_1 \oplus \cdots \oplus \mathcal{H}_n$ with the norm $|||(f_i)_{i=1}^n|||_1 = \sum_{i=1}^n \|f_i\|_i$, and denote the corresponding operator norm by $|||\cdot|||_1$ as well, then

$$|||\mathcal{C}(\lambda)|||_1 \leq \max_{j=1}^{n} \sum_{\substack{i=1\\ i\neq j}}^{n} \|A_{ij}(A_{jj}-\lambda)^{-1}\|.$$

If $\lambda \notin \mathcal{G}_{\mathrm{row}}(\mathcal{A})$, then $\sum_{\substack{i=1\\ i\neq j}}^{n} \big\|A_{ij}(A_{jj}-\lambda)^{-1}\big\| < 1$ for all $j = 1, 2, \ldots, n$, and thus, $|||\mathcal{C}(\lambda)|||_1 < 1$. By (6.2), this implies $\lambda \in \rho(\mathcal{A})$, which proves the inclusion $\sigma(\mathcal{A}) \subset \mathcal{G}_{\mathrm{row}}(\mathcal{A})$.

The relative bounds δ_{ij} are defined as the infima of all $b_{ij} \geq 0$ such that there exists an $a_{ij} \geq 0$ with (6.1) (compare to (2.4)). Hence, the assumption $\sum_{\substack{j=1\\ j\neq i}}^{n} \delta_{ij} < 1$ implies the existence of a_{ij}, $b_{ij} \geq 0$ with $\sum_{\substack{j=1\\ j\neq i}}^{n} b_{ij} < 1$ and (6.1). Now, the inclusion for $\mathcal{G}_{\mathrm{row}}(\mathcal{A})$ follows if we use that, by (6.1),

$$\|A_{ij}(A_{jj}-\lambda)^{-1}\| \leq (a_{ij} + |\lambda| b_{ij})\|(A_{jj}-\lambda)^{-1}\| + b_{ij}$$

for $i, j = 1, 2, \ldots, n,\ i \neq j$. □

Theorem 6.4 (Column sum Gershgorin theorem)**.** *Let $\mathcal{A}$ be a diagonally dominant $n\times n$ operator matrix with closed diagonal elements A_{jj} and A_{jj}-bounds δ_{ij} of A_{ij} such that $\sum_{\substack{j=1\\ j\neq i}}^{n} \delta_{ij} < 1$, and let a_{ij}, $b_{ij} \geq 0$ be such that $\sum_{\substack{j=1\\ j\neq i}}^{n} b_{ij} < 1$ and*

$$\|A_{ij}f_j\|_i \leq a_{ij}\|f_j\|_j + b_{ij}\|A_{jj}f_j\|_j, \quad f_j \in D(A_{jj}) \subset D(A_{ij}),$$

for $i, j = 1, 2, \ldots, n,\ i \neq j$. Then, $\sigma(\mathcal{A}) \subset \mathcal{G}_{\mathrm{col}}(\mathcal{A})$ where

$$\mathcal{G}_{\mathrm{col}}(\mathcal{A}) := \bigcup_{i=1}^{n}\left(\sigma(A_{ii}) \cup \left\{\lambda \in \rho(A_{ii}) : \sum_{\substack{j=1\\ j\neq i}}^{n} \|A_{ij}(A_{jj}-\lambda)^{-1}\| \geq 1\right\}\right)$$

$$\subset \bigcup_{j=1}^{n}\left(\sigma(A_{jj}) \cup \left\{\lambda \in \rho(A_{jj}) : \sum_{\substack{j=1\\ j\neq i}}^{n} \|(A_{jj}-\lambda)^{-1}\|(a_{ij} + |\lambda| b_{ij}) \geq 1 - \sum_{\substack{j=1\\ j\neq i}}^{n} b_{ij}\right\}\right).$$

Proof. The proof is analogous to that of Theorem 6.3 if we equip $\mathcal{H} = \mathcal{H}_1 \oplus \cdots \oplus \mathcal{H}_n$ with the norm $|||(f_i)_{i=1}^n|||_\infty = \max_{i=1}^n \|f_i\|_i$, denote the corresponding operator norm by $|||\cdot|||_\infty$ and use the estimate

$$|||\mathcal{C}(\lambda)|||_\infty \le \max_{i=1}^n \sum_{\substack{j=1\\ j\neq i}}^n \|A_{ij}(A_{jj}-\lambda)^{-1}\|. \qquad \square$$

If the off-diagonal entries A_{ij}, $i \neq j$, of $\mathcal{A}$ are bounded, we can choose $a_{ij} = \|A_{ij}\|$, $b_{ij} = 0$, $i \neq j$, in Theorems 6.3 and 6.4. For the set $\mathcal{G}_{\mathrm{col}}(\mathcal{A})$ in Theorem 6.4 we obtain the corresponding inclusion merely by replacing $\|A_{ij}(A_{jj}-\lambda)^{-1}\|$ by $\|A_{ij}\|\|(A_{jj}-\lambda)^{-1}\|$ and is thus omitted. For the set $\mathcal{G}_{\mathrm{row}}(\mathcal{A})$ in Theorem 6.3 we obtain the following inclusion.

Corollary 6.5. *Let $\mathcal{A}$ be an $n \times n$ operator matrix as in Theorem 6.3 with bounded off-diagonal entries. Then,*

$$\mathcal{G}_{\mathrm{row}}(\mathcal{A}) \subset \bigcup_{j=1}^n \left(\sigma(A_{jj}) \cup \left\{\lambda \in \rho(A_{jj}) : \|(A_{jj}-\lambda)^{-1}\|^{-1} \le \sum_{\substack{i=1\\ i\neq j}}^n \|A_{ij}\|\right\}\right).$$

The disadvantage of using Gershgorin-type theorems is that they involve the norms of inverses of the diagonal entries A_{jj}. If we estimate the latter in terms of the numerical ranges of A_{jj}, the following enclosures are immediate from Theorem 6.3 and Corollary 6.5.

Corollary 6.6. *Let $\mathcal{A}$ be a diagonally dominant $n \times n$ operator matrix as in Theorem 6.3 and a_{ij}, b_{ij} as in (6.1). Then,*

$$\mathcal{G}_{\mathrm{row}}(\mathcal{A}) \subset \bigcup_{j=1}^n \left\{\lambda \in \mathbb{C} : \operatorname{dist}(\lambda, W(A_{jj})) \le \frac{\sum_{\substack{i=1\\ i\neq j}}^n (a_{ij} + |\lambda| b_{ij})}{1 - \sum_{\substack{i=1\\ i\neq j}}^n b_{ij}}\right\};$$

if the off-diagonal entries A_{ij}, $i \neq j$, are bounded, then

$$\mathcal{G}_{\mathrm{row}}(\mathcal{A}) \subset \bigcup_{j=1}^n \left\{\lambda \in \mathbb{C} : \operatorname{dist}(\lambda, W(A_{jj})) \le \sum_{\substack{i=1\\ i\neq j}}^n \|A_{ij}\|\right\}. \tag{6.3}$$

Remark 6.7. The set on the right hand side of (6.3) also contains the block numerical range of $\mathcal{A}$,

$$W^n(\mathcal{A}) \subset \bigcup_{j=1}^{n} \left\{ \lambda \in \mathbb{C} : \operatorname{dist}(\lambda, W(A_{jj})) \leq \sum_{\substack{i=1 \\ i \neq j}}^{n} \|A_{ij}\| \right\};$$

this follows if we apply the matrix Gershgorin theorem to each matrix $\mathcal{A}_f$, $f = (f_j)_{j=1}^n \in D(\mathcal{A}) \cap \mathbb{S}^n$, in Definition 4.1 of $W^n(\mathcal{A})$ and use that $\operatorname{dist}(\lambda, W(A_{jj})) \leq |(A_{jj}f_j, f_j)_j - \lambda|$ and $|(A_{ij}f_j, f_i)_i| \leq \|A_{ij}\|$, $i, j = 1, 2, \ldots, n$, $i \neq j$.

In the next section, we show that, for self-adjoint tridiagonal 3×3 operator matrices, the block numerical range indeed yields tighter estimates than, e.g., the row sum Gershgorin theorem.

7. Self-adjoint $n \times n$ operator matrices and applications.

In this section, we show that the block numerical range may give tighter spectral enclosures than the numerical range, classical perturbation theory and Gershgorin-type enclosures. It is remarkable that this effect occurs even for self-adjoint operator matrices.

Here, we consider the cubic numerical range ($n = 3$) of self-adjoint tridiagonal operator matrices with semi-bounded diagonal entries and bounded off-diagonal entries. Toward this end, we need the following elementary lemma on the eigenvalues of 3×3 matrices.

Lemma 7.1. *Let $a_{ii} \in \mathbb{R}$, $a_{ij} \in \mathbb{C}$, $i \neq j$, $i, j = 1, 2, 3$, and consider the Hermitian tridiagonal 3×3 matrix*

$$M := \begin{pmatrix} a_{11} & a_{12} & 0 \\ \overline{a_{12}} & a_{22} & a_{23} \\ 0 & \overline{a_{23}} & a_{33} \end{pmatrix}.$$

Then, the eigenvalues $\lambda_1 \leq \lambda_2 \leq \lambda_3$ of M have the following properties:

(i) *λ_k, $k = 1, 2, 3$, depends only upon $a_{ii} \in \mathbb{R}$, $i = 1, 2, 3$, and upon $|a_{12}|$, $|a_{23}| \in \mathbb{R}$;*

(ii) *for fixed $a_{12}, a_{23} \in \mathbb{C}$, λ_k, $k = 1, 2, 3$, is a monotonically increasing function of each $a_{ii} \in \mathbb{R}$, $i \in \{1, 2, 3\}$;*

(ii′) *for fixed* $a_{12}, a_{23} \in \mathbb{C}$, λ_k, $k = 1, 2, 3$, *is a strictly monotonically increasing function of* $a_{ii} \in \mathbb{R}$, $i \in \{1, 2, 3\}$, *provided that* λ_k *is not an eigenvalue of any principal minor of* M *not containing the* i*th row and column*;

(iii) *for fixed* $a_{ii} \in \mathbb{R}$, $i = 1, 2, 3$, λ_1 (*respectively,* λ_3) *is a monotonically decreasing* (*respectively, increasing*) *function of* $|a_{12}|$, $|a_{23}|$, *separately*;

(iv) λ_k, $k = 1, 2, 3$, *are given by the formulae*:

$$\lambda_k = \frac{1}{3}(a_{11} + a_{22} + a_{33}) \quad \text{if} \;\; a_{11} = a_{22} = a_{33},\; a_{12} = a_{23} = 0,$$

$$\lambda_k = \frac{1}{3}(a_{11} + a_{22} + a_{33}) + 2\sqrt{-\frac{p}{3}} \cos\frac{\varphi + 2k\pi}{3} \quad \text{otherwise}, \tag{7.1}$$

where, in the latter case,

$$p := -\frac{1}{6}\big((a_{11} - a_{22})^2 + (a_{11} - a_{33})^2 + (a_{22} - a_{33})^2\big) - |a_{12}|^2 - |a_{23}|^2 \;\; (< 0), \tag{7.2}$$

$$q := |a_{12}|^2 a_{33} + |a_{23}|^2 a_{11} - a_{11}a_{22}a_{33} - \frac{2}{27}(a_{11} + a_{22} + a_{33})^3 + \frac{1}{3}(a_{11}a_{22} + a_{11}a_{33} + a_{22}a_{33} - |a_{12}|^2 - |a_{23}|^2) \cdot (a_{11} + a_{22} + a_{33}), \tag{7.3}$$

$$\varphi := \arccos\left(-\frac{3q}{2p}\sqrt{-\frac{3}{p}}\right); \tag{7.4}$$

(v) *if* $a_{11} = a_{22} = a_{33} = \min_{i=1}^{3} a_{ii}$, *then* $q = 0$, *and hence,*

$$\lambda_k \geq \min_{i=1}^{3} a_{ii} - \sqrt{|a_{12}|^2 + |a_{23}|^2}, \quad k = 1, 2, 3; \tag{7.5}$$

if at least two of the a_{ii}, $i = 1, 2, 3$, *are different and* λ_k *is not an eigenvalue of any proper principal minor of* M, *then the inequality in* (7.5) *is strict.*

Proof.

(i) Since the eigenvalue equation

$$\begin{aligned} &\lambda^3 - \big(a_{11} + a_{22} + a_{33}\big)\lambda^2 + \big(a_{11}a_{22} + a_{11}a_{33} + a_{22}a_{33}\big)\lambda \\ &- a_{11}a_{22}a_{33} = \big(|a_{12}|^2 + |a_{23}|^2\big)\lambda - |a_{12}|^2 a_{33} - |a_{23}|^2 a_{11} \end{aligned} \tag{7.6}$$

for M depends only upon $a_{ii} \in \mathbb{R}$, $i = 1, 2, 3$, and $|a_{12}|$, $|a_{23}| \in \mathbb{R}$, claim (i) is obvious.

(ii) By the classical min-max principle, the eigenvalues λ_k of M, $k = 1, 2, 3$, can be characterized as

$$\begin{aligned} \lambda_1 &= \min_{x \in \mathbb{S}^3} (Mx, x), \\ \lambda_2 &= \max_{y \in \mathbb{S}^3} \min_{x \cap \mathbb{S}^3, x \perp y} (Mx, x), \\ \lambda_3 &= \max_{x \in \mathbb{S}^3} (Mx, x). \end{aligned}$$

Since, for $x = (x_1, x_2, x_3) \in \mathbb{S}^3$, the quadratic form

$$\begin{aligned} (Mx, x) =& a_{11}|x_1|^2 + a_{22}|x_2|^2 + a_{33}|x_3|^2 \\ &+ 2\,\mathrm{Re}(a_{12}x_2\overline{x_1}) + 2\,\mathrm{Re}(a_{23}x_3\overline{x_2}) \end{aligned} \tag{7.7}$$

is a monotonically increasing function with respect to a_{ii}, $i \in \{1, 2, 3\}$, claim (ii) follows.

(ii′) We denote by $M_{lm} \in M_2(\mathbb{C})$ the principal minor of M consisting of the lth and mth row and column, and set $M_l = (a_{ll}) \in M_1(\mathbb{R})$, $l = 1, 2, 3$.

Let $k, i \in \{1, 2, 3\}$, and let $x^k = (x_1^k, x_2^k, x_3^k)^{\mathrm{t}} \in \mathbb{C}^3$, $\|x^k\| = 1$, be an eigenvector of M at λ_k. Then, (7.7) shows that $\lambda_k = (Mx^k, x^k)$ is strictly increasing with respect to a_{ii} if $x_i^k \neq 0$. In order to prove the claim, we show that the assumption implies that $x_i^k \neq 0$. First, let $i = 1$ and assume, to the contrary, that $x_1^k = 0$. Then, the eigenvalue equation $Mx^k = \lambda_k x^k$ becomes

$$\begin{pmatrix} a_{12}x_2^k \\ a_{22}x_2^k + a_{23}x_3^k \\ \overline{a_{23}}x_2^k + a_{33}x_3^k \end{pmatrix} = \lambda_k \begin{pmatrix} 0 \\ x_2^k \\ x_3^k \end{pmatrix}.$$

Then, either $a_{12} = 0$ and λ_k is an eigenvalue of M_{23} or $x_2^k = 0$ and $\lambda_k = a_{33}$ is an eigenvalue of M_3, a contradiction to the assumption.

For $i=3$, the proof is analogous; here, $x_3^k=0$ yields the contradiction $a_{23}=0$ and λ_k is an eigenvalue of M_{12} or $\lambda_k=a_{11}$ is an eigenvalue of M_1.

For $i=2$, the assumption $x_2^k=0$ implies that

$$\begin{pmatrix} a_{11}x_1^k \\ \overline{a_{12}}x_1^k + a_{23}x_3^k \\ a_{33}x_3^k \end{pmatrix} = \lambda_k \begin{pmatrix} x_1^k \\ 0 \\ x_3^k \end{pmatrix}.$$

Since $\|x^k\|=1$, either $x_1^k \neq 0$ or $x_3^k \neq 0$, and hence, either $\lambda_k = a_{11}$ is an eigenvalue of M_1 or $\lambda_k = a_{33}$ is an eigenvalue of M_3, both of which contradict the assumption.

(iii) We fix $a_{ii} \in \mathbb{R}$, $i=1,2,3$, and $|a_{23}|$; the proof for fixed $|a_{12}|$ is completely analogous. If we denote the right hand side of the eigenvalue equation (7.6) by $g(\lambda, |a_{12}|)$, then the eigenvalues λ_k, $k=1,2,3$, as functions of $|a_{12}|$ can be viewed as the intersection points of a fixed monic cubic polynomial with the family of linear polynomials $g(\cdot, |a_{12}|)$, $|a_{12}| \in [0,\infty)$,

$$(\lambda - a_{11})(\lambda - a_{22})(\lambda - a_{33}) = g(\lambda, |a_{12}|), \quad \lambda \in \mathbb{R}.$$

The family of lines $g(\cdot, |a_{12}|)$, $|a_{12}| \in [0,\infty)$, intersects in the point $\lambda = a_{33}$ and has the common value $|a_{23}|^2(a_{33}-a_{11})$ there; at the point $\lambda = a_{11}$ the value is $|a_{12}|^2(a_{11}-a_{33})$. Hence, all of these lines are non-positive at $\min\{a_{11}, a_{33}\}$ and non-negative at $\max\{a_{11}, a_{33}\}$. Since the slopes $\left(|a_{12}|^2 + |a_{23}|^2\right)$ are monotonically increasing in $|a_{12}| \in [0,\infty)$, the claimed monotonicity of λ_1 and λ_3 with respect to $|a_{12}| \in [0,\infty)$ follows.

(iv) By the change of variables

$$x = \lambda - \frac{1}{3}(a_{11}+a_{22}+a_{33}), \tag{7.8}$$

(7.6) is reduced to the depressed cubic equation $x^3+px+q=0$ with p, q given by (7.2) and (7.3), respectively; note that $p \le 0$. Since the matrix M is Hermitian, it has only real eigenvalues $\lambda_1 \le \lambda_2 \le \lambda_3$. This is the case of non-positive discriminant, $27q^2+4q^3 \le 0$, and hence, the roots of the depressed equation are given by

$$2x_k = 0, \qquad\qquad k = 1, 2, 3, \quad \text{if } p = 0;$$
$$x_k = 2\sqrt{-\frac{p}{3}}\cos\frac{\varphi + 2k\pi}{3}, \qquad\qquad k = 1, 2, 3, \quad \text{if } p < 0,$$

(see [**4,** Theorem 1.3.3]; note that we changed the order of the roots to achieve that they are enumerated increasingly, i.e., $x_1 \le x_2 \le x_3$. This and (7.8) yield the formulae for λ_k, $k = 1, 2, 3$.

(v) Clearly, if $a_{11} = a_{22} = a_{33}$, then $p = -(|a_{12}|^2 + |a_{23}|^2)$ and $q = 0$. Hence, $\varphi = \pi/2$ and $\cos(((\pi/2) + 2\pi)/3) = -\sqrt{3}/2$ which implies $\lambda_1 \ge \min_{i=1}^{3} a_{ii} - \sqrt{-p}$. The last claim follows if we use the strict monotonicity from (ii′). $\square$

In the following theorem, we consider self-adjoint tridiagonal 3×3 operator matrices whose diagonal entries are all semi-bounded either from below or from above.

We assume that both off-diagonal entries A_{12}, A_{23} are non-zero; otherwise, the cubic numerical range is simply the union of a quadratic numerical range and a numerical range. For simplicity, we assume that the off-diagonal entries are bounded.

Theorem 7.2. *Let*

$$\mathcal{A} = \begin{pmatrix} A_{11} & A_{12} & 0 \\ A_{12}^* & A_{22} & A_{23} \\ 0 & A_{23}^* & A_{33} \end{pmatrix} = \underbrace{\begin{pmatrix} A_{11} & 0 & 0 \\ 0 & A_{22} & 0 \\ 0 & 0 & A_{33} \end{pmatrix}}_{=\mathcal{T}} + \underbrace{\begin{pmatrix} 0 & A_{12} & 0 \\ A_{12}^* & 0 & A_{23} \\ 0 & A_{23}^* & 0 \end{pmatrix}}_{\mathcal{S}}$$

with $A_{ii} = A_{ii}^*$, $i = 1, 2, 3$, all bounded from below and bounded A_{12}, $A_{23} \ne 0$. Then, $\mathcal{A} = \mathcal{A}^*$ is bounded from below with

$$\min\sigma(\mathcal{A}) = \inf W^3(\mathcal{A}) \ge w_-^3(\mathcal{A}) := \frac{1}{3}\sum_{i=1}^{3} a_i + 2\sqrt{-\frac{p_0}{3}}\cos\frac{\varphi_0 + 2\pi}{3} \tag{7.9}$$

where $a_i := \min\sigma(A_{ii})$, $i = 1, 2, 3$, and

$$p_0 := -\frac{1}{6}\sum_{\substack{i,j=1 \\ i<j}}^{3} (a_i - a_j)^2 - \|A_{12}\|^2 - \|A_{23}\|^2;$$

$$
\begin{aligned}
q_0 &:= \|A_{12}\|^2 a_3 + \|A_{23}\|^2 a_1 - a_1 a_2 a_3 - \frac{2}{27}(a_1+a_2+a_3)^3 \\
&\quad + \frac{1}{3}(a_1 a_2 + a_1 a_3 + a_2 a_3 - \|A_{12}\|^2 - \|A_{23}\|^2)(a_1+a_2+a_3); \\
\varphi_0 &:= \arccos\left(-\frac{3q_0}{2p_0}\sqrt{-\frac{3}{p_0}}\right).
\end{aligned}
$$

Moreover, the lower bound $w_-^3(\mathcal{A})$ of $W^3(\mathcal{A})$ satisfies the inequality

$$
w_-^3(\mathcal{A}) \geq \min\sigma(\mathcal{T}) - (\|A_{12}\|^2 + \|A_{23}\|^2)^{1/2}, \tag{7.10}
$$

and strict inequality prevails if at least two of the lower bounds $\min\sigma(A_{ii})$, $i=1,2,3$, of the diagonal elements are different.

Remark 7.3.

(i) The estimate (7.9) also holds for any a_i with $A_{ii} \geq a_i$, $i=1,2,3$.

(ii) If, in Theorem 7.2, all $A_{ii} = A_{ii}^*$, $i=1,2,3$, are bounded from above, an analogous estimate holds for $\max\sigma(\mathcal{A}) \leq w_+^3(\mathcal{A})$ where $w_+^3(\mathcal{A})$ is given by (7.9) with $a_i := \max\sigma(A_{ii})$, $i=1,2,3$.

Proof. Let $f=(f_1,f_2,f_3) \in D(\mathcal{A}) \cap \mathbb{S}^3$, i.e., $\|f_i\|_i = 1$, $i=1,2,3$, and consider

$$
\mathcal{A}_f = \Big((A_{ij} f_j, f_i)_i \Big)_{i,j=1}^3 = \begin{pmatrix} (A_{11}f_1,f_1)_1 & (A_{12}f_2,f_1)_1 & 0 \\ (A_{12}^* f_1, f_2)_2 & (A_{22}f_2,f_2)_2 & (A_{23}f_3,f_2)_2 \\ 0 & (A_{23}^* f_2, f_3)_3 & (A_{33}f_3,f_3)_3 \end{pmatrix}.
$$

By Lemma 7.1, $\mathcal{A}_f$ has three eigenvalues $\lambda_1(f) \leq \lambda_2(f) \leq \lambda_3(f)$ where $\lambda_1(f)$ is a monotonically increasing function of $(A_{ii}f_i,f_i)_i \in \mathbb{R}$, $i=1,2,3$, separately and a monotonically decreasing function of $|(A_{12}f_2,f_1)_1|$, $|(A_{23}f_3,f_2)_2|$ separately. If we use this monotonicity together with the inequalities

$$
(A_{ii}f_i,f_i)_i \geq a_i \geq \min\sigma(\mathcal{T}), \quad i=1,2,3, \tag{7.11}
$$

$$
\begin{aligned}
|(A_{12}f_2,f_1)_1|,\ |(A_{12}^*f_1,f_2)_2| &\leq \|A_{12}\|, \\
|(A_{23}f_3,f_2)_2|,\ |(A_{23}^*f_2,f_3)_3| &\leq \|A_{23}\|,
\end{aligned} \tag{7.12}
$$

in the formulae for $\lambda_k(f)$ according to Lemma 7.1 (iv) and (v), then the first inequality in (7.11), yields that

$$\inf W^3(\mathcal{A}) = \inf_{f \in D(\mathcal{A}) \cap \mathbb{S}^3} \lambda_1(f) \geq \frac{1}{3} \sum_{i=1}^{3} a_i + 2\sqrt{-\frac{p_0}{3}} \cos \frac{\varphi_0 + 2\pi}{3},$$

while the second inequality in (7.11) with equal lower bound for $(A_{ii} f_i, f_i)_i$, $i = 1, 2, 3$, together with (7.5), gives (7.10).

Since $\mathcal{A}$ is self-adjoint and semi-bounded, it follows that $W^3(\mathcal{A}) \subset \mathbb{R}$, $\mathbb{C} \setminus W^3(\mathcal{A})$ has one component, and $\mathbb{C} \setminus \mathbb{R} \subset \rho(\mathcal{A})$. Thus, Theorem 5.2 shows that $\sigma(\mathcal{A}) \subset \overline{W^3(\mathcal{A})}$, and hence, $\min \sigma(\mathcal{A}) \geq \inf W^3(\mathcal{A})$, which completes the proof of (7.10).

Finally, suppose that at least two of the $a_i = \min \sigma(A_{ii})$, $i = 1, 2, 3$, are different, $\min \sigma(A_{i_0 i_0}) > \min \sigma(\mathcal{T})$ for some $i_0 \in \{1, 2, 3\}$. Since the strict monotonicity in Lemma (7.1) (ii$'$) only holds under additional assumptions, we distinguish two cases.

First, assume that $\inf W^3(\mathcal{A}) < \inf W^{\#\mathcal{I}}(\mathcal{A}_{\mathcal{I}})$ for all proper principal minors $\mathcal{A}_{\mathcal{I}}$ of $\mathcal{A}$ with $\mathcal{I} \subsetneq \{1, 2, 3\}$, and let $\varepsilon > 0$ with $\inf W^3(\mathcal{A}) + \varepsilon < \inf W^{\#\mathcal{I}}(\mathcal{A}_{\mathcal{I}})$ for all $\mathcal{I} \subsetneq \{1, 2, 3\}$ be arbitrary. Then, there exists an $f^\varepsilon = (f_1^\varepsilon, f_2^\varepsilon, f_3^\varepsilon) \in D(\mathcal{A}) \cap \mathbb{S}^3$ such that $\lambda_1(f^\varepsilon) < \inf W^3(\mathcal{A}) + \varepsilon < \inf W^{\#\mathcal{I}}(\mathcal{A}_{\mathcal{I}})$ for all $\mathcal{I} \subsetneq \{1, 2, 3\}$. Hence, $\lambda_1(f^\varepsilon)$ is not an eigenvalue of any proper principal minor of $\mathcal{A}_{f^\varepsilon}$. According to Lemma 7.1 (ii$'$), $\lambda_1(f^\varepsilon)$ is strictly monotonically increasing with respect to $(A_{i_0 i_0} f_{i_0}^\varepsilon, f_{i_0}^\varepsilon)_{i_0}$. Now, the strict inequality $(A_{i_0 i_0} f_{i_0}^\varepsilon, f_{i_0}^\varepsilon)_{i_0} \geq \min \sigma(A_{i_0 i_0}) > \min \sigma(\mathcal{T})$ and the inequalities (7.11) for $i \in \{1, 2, 3\} \setminus \{i_0\}$ yield that strict inequality in (7.10) holds.

Finally, suppose that $\inf W^3(\mathcal{A}) = \inf W^{\#\mathcal{I}}(\mathcal{A}_{\mathcal{I}})$ for some proper principal minor $\mathcal{A}_{\mathcal{I}}$ of $\mathcal{A}$ with $\mathcal{I} \subsetneq \{1, 2, 3, \}$. If $\mathcal{I} = \{i\}$ with $i \in \{1, 2, 3\}$, then $\inf W^3(\mathcal{A}) = \inf W(A_{ii}) = \min \sigma(A_{ii}) \geq \min \sigma(\mathcal{T}) > \min \sigma(\mathcal{T}) - (\|A_{12}\|^2 + \|A_{23}\|^2)^{1/2}$ since A_{12}, $A_{23} \neq 0$ by assumption. If $\mathcal{I} = \{1, 3\}$, then $\inf W^3(\mathcal{A}) = \inf W^2(\operatorname{diag}(A_{11}, A_{33})) \geq \min \sigma(\mathcal{T}) > \min \sigma(\mathcal{T}) - (\|A_{12}\|^2 + \|A_{23}\|^2)^{1/2}$, as before. If $\mathcal{I} = \{1, 2\}$, then classical perturbation theory for $\mathcal{A}_{12}$ shows that $\inf W^3(\mathcal{A}) = \inf W^2(\mathcal{A}_{12}) \geq \min_{i=1,2} \sigma(A_{ii}) - \|A_{12}\| > \min \sigma(\mathcal{T}) - (\|A_{12}\|^2 + \|A_{23}\|^2)^{1/2}$ since $A_{23} \neq 0$ by assumption; the case $\mathcal{I} = \{2, 3\}$ is analogous. □

Remark 7.4. Let $\mathcal{A}$ be a self-adjoint $n \times n$ operator matrix in $\mathcal{H} = \mathcal{H}_1 \oplus \cdots \oplus \mathcal{H}_n$ with diagonal elements A_{ii} all bounded from below and bounded off-diagonal entries.

(i) The row sum Gershgorin enclosure in Corollary 6.5 yields the lower bound

$$\min \sigma(\mathcal{A}) \geq \inf \mathcal{G}_{\mathrm{row}}(\mathcal{A}) \geq g_{\mathrm{row},-}(\mathcal{A}) := \min_{i=1}^{n} \left(\min \sigma(A_{ii}) - \sum_{\substack{j=1 \\ i \neq j}}^{n} \|A_{ij}\| \right)$$

since here, for $i = 1, 2, \dots, n$,

$$\|(A_{ii} - \lambda)^{-1}\| = \frac{1}{\min \sigma(A_{ii}) - \lambda}, \qquad \lambda < \min \sigma(\mathcal{T}) \leq \min \sigma(A_{ii});$$

analogous bounds can be derived for $\max \sigma(\mathcal{A})$.

(ii) Classical perturbation theory, more precisely, a Neumann series argument, gives the lower bound

$$\min \sigma(\mathcal{A}) \geq p_-(\mathcal{A}) := \min_{i=1}^{n} (\min \sigma(A_{ii})) - \|\mathcal{S}\|$$

since, for $\lambda < p_-(\mathcal{A})$, we have $\mathcal{A} - \lambda = (\mathcal{T} - \lambda)(I_{I_{\mathcal{H}}} + \mathcal{S}(\mathcal{T} - \lambda)^{-1})$ and

$$\|\mathcal{S}(\mathcal{T} - \lambda)^{-1}\| \leq \frac{\|\mathcal{S}\|}{\operatorname{dist}(\lambda, \sigma(\mathcal{T}))} = \frac{\|\mathcal{S}\|}{\min_{i=1}^{n} (\min \sigma(A_{ii})) - \lambda} < 1.$$

Remark 7.5. If, in Theorem 7.2, all A_{ii} have the same lower bound, then the lower bound $w_-^3(\mathcal{A})$ in (7.9) satisfies (7.10),

$$w_-^3(\mathcal{A}) \geq \min \sigma(\mathcal{T}) - (\|A_{12}\|^2 + \|A_{23}\|^2)^{1/2};$$

the same estimate holds for the perturbation bound in Remark 7.4 (ii),

$$p_-(\mathcal{A}) = \min \sigma(\mathcal{T}) - \|\mathcal{S}\| \geq \min \sigma(\mathcal{T}) - \left(\|A_{12}\|^2 + \|A_{23}\|^2\right)^{1/2}.$$

The next example shows that the lower bound for the spectrum provided by the block numerical range is tighter than the bounds furnished by the Gershgorin enclosures and by classical perturbation theory.

Example 7.6. Let $a \geq 0$, and consider the 3×3 matrix

$$\mathcal{A} = (a_{ij})_{i,j=1}^{3} = \begin{pmatrix} a & -1 & 0 \\ -1 & 0 & 1 \\ 0 & 1 & -a \end{pmatrix} = \underbrace{\begin{pmatrix} a & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -a \end{pmatrix}}_{\mathcal{T}} + \underbrace{\begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}}_{\mathcal{S}}.$$

Here, $W^3(\mathcal{A}) = \sigma_p(\mathcal{A})$, and thus, $\min W^3(\mathcal{A}) = -\sqrt{a^2+2}$. With the notation of Theorem 7.2, we have $a_1 = a$, $a_2 = 0$, $a_3 = -a$, $\|A_{12}\| = \|A_{23}\| = 1$, $p_0 = -a^2 - 2$, $q_0 = 0$, $\varphi_0 = -\pi/2$. Hence, the lower bound $w^3_-(\mathcal{A})$ in Theorem 7.2 becomes

$$w^3_-(\mathcal{A}) = \frac{2\sqrt{a^2+2}}{\sqrt{3}} \cos\left(\frac{5\pi}{6}\right) = -\sqrt{a^2+2} = \min W^3(\mathcal{A}) = \min \sigma(\mathcal{A}),$$

and is thus sharp. This also follows from the proof of Theorem 7.2 since all $\mathcal{H}_i$, $i = 1, 2, 3$, have dimension 1, and thus, the estimates (7.11), (7.12) become equalities.

Gershgorin's row sum theorem for matrices (see **[7,** Theorem 6.1.1] or Remark 7.4 with $\mathcal{H}_i = \mathbb{C}$) yields the lower bound

$$\begin{aligned} g_{\mathrm{row},-}(\mathcal{A}) = \min_{i=1}^{3} \Bigg(a_{ii} - \sum_{\substack{j=1 \\ i\neq j}}^{3} |a_{ij}| \Bigg) &= \min\{a-1, -2, -a-1\} \\ &= \begin{cases} -2 & a \le 1, \\ -a-1 & a \ge 1, \end{cases} \end{aligned}$$

and classical perturbation theory with respect to the diagonal gives the lower bound

$$p_-(\mathcal{A}) = \min \sigma(\mathcal{T}) - \|\mathcal{S}\| = -a - \sqrt{2}. \tag{7.13}$$

The only parameter value for which the three different bounds coincide is $a = 0$ where all diagonal elements are equal. For all $a > 0$, both the Gershgorin and the perturbation bound are worse than the bound $w^3_-(\mathcal{A})$ provided by the cubic numerical range (see Figure 1).

As an example, we apply Theorem 7.2 to three-channel Hamiltonians arising in non-relativistic quantum mechanics.

Example 7.7. A simple model of interaction between two confined channels (e.g., a quark/anti-quark system) and a scattering channel (e.g., a two-hadron system) leads to a three-channel Hamiltonian of the form

$$\mathbf{H} := \begin{pmatrix} -\nabla^2/2 + U_1 & V_{12} & 0 \\ \overline{V_{12}} & -\nabla^2/2 + U_2 & V_{23} \\ 0 & \overline{V_{23}} & -\nabla^2/2 \end{pmatrix}$$

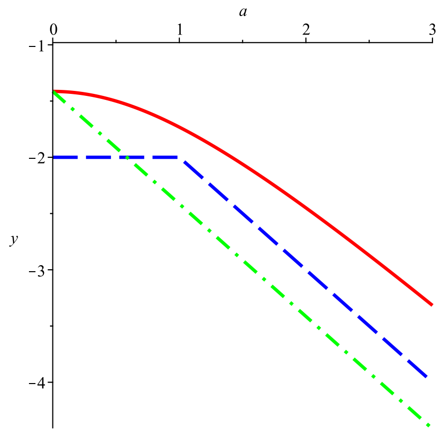

FIGURE 1. Lower bounds $w^3_-(\mathcal{A})$ (red/solid, cubic numerical range), $g_{\mathrm{row},-}(\mathcal{A})$ (blue/dashed, Gershgorin), and $p_-(\mathcal{A})$ (green/dash-dotted, perturbation theory) as functions of a.

in the Hilbert space $L_2(\mathbb{R}^3) \oplus L_2(\mathbb{R}^3) \oplus L_2(\mathbb{R}^3)$; here $\hbar$, the quark mass, and the hadronic ground state mass have been normalized to unity. We decompose $\mathbf{H} = \mathbf{H}_0 + \mathbf{V}$ with

$$\mathbf{H}_0 := \begin{pmatrix} H_{\mathrm{c}}^{(1)} & 0 & 0 \\ 0 & H_{\mathrm{c}}^{(2)} & 0 \\ 0 & 0 & H_{\mathrm{s}} \end{pmatrix}, \qquad \mathbf{V} := \begin{pmatrix} 0 & V_{12} & 0 \\ \overline{V_{12}} & 0 & V_{23} \\ 0 & \overline{V_{23}} & 0 \end{pmatrix}.$$

For the Hamiltonians $H_{\mathrm{c}}^{(i)} = -\nabla^2/2 + U_i$, $i = 1, 2$, in the confined channels, we assume that the potential U_i is such that $H_{\mathrm{c}}^{(i)}$ is self-adjoint in $L_2(\mathbb{R}^3)$ and bounded from below with discrete spectrum, $H_{\mathrm{c}}^{(i)} \geq -\omega_i$ where $\omega_i > 0$,

$$\sigma(H_{\mathrm{c}}^{(i)}) = \sigma_{\mathrm{p}}(H_{\mathrm{c}}^{(i)}) = \{\mu_n^{(i)} : n \in \mathbb{N}\} \subset [-\omega_i, \infty)$$

accumulating only at ∞; this includes, e.g., three-dimensional harmonic oscillators.

The unperturbed Hamiltonian $H_{\mathrm{s}} = -\nabla^2/2$ in the scattering channel, which is the kinetic energy operator for the relative motion between the two hadrons, is self-adjoint and non-negative in $L_2(\mathbb{R}^3)$ with continuous spectrum, $\sigma(H_{\mathrm{s}}) = \sigma_{\mathrm{c}}(H_{\mathrm{s}}) = [0, \infty)$.

The off-diagonal terms V_{ij} and $V_{ji} = \overline{V_{ij}}$, $|i-j| = 1$, $i, j = 1, 2, 3$, represent the coupling between the channels. We suppose that $V_{21} = \overline{V_{12}}$ is $H_{\mathrm{c}}^{(1)}$-bounded with relative bound δ_{21}, V_{12} and $V_{32} = \overline{V_{23}}$ are $H_{\mathrm{c}}^{(2)}$-bounded with relative bounds δ_{12} and δ_{32}, respectively, and V_{23} is H_{s}-bounded with relative bound δ_{23}. If

$$\sqrt{2}\max\left\{\delta_{21}, \sqrt{\delta_{12}^2 + \delta_{32}^2}, \delta_{32}\right\} < 1,$$

then $\mathbf{H}$ is diagonally dominant of order < 1, whence self-adjoint on the domain

$$D(\mathbf{H}) = D(H_{\mathrm{c}}^{(1)}) \oplus D(H_{\mathrm{c}}^{(2)}) \oplus D(H_{\mathrm{s}})$$

and bounded from below. If the potentials V_{ij} are essentially bounded, $V_{ij} \in L_\infty(\mathbb{R}^3)$, then Theorem 7.2 and Remark 7.4 give the following lower bounds for the spectrum of $\mathbf{H}$.

Since ω_1, $\omega_2 > 0$, Theorem 7.2 shows that

$$\begin{aligned}\min\sigma(\mathbf{H}) \geq w_-^3(\mathbf{H}) &= -\frac{1}{3}(\omega_1+\omega_2) + 2\sqrt{-\frac{p_0}{3}}\cos\frac{\varphi_0+2\pi}{3}\\ &> \min\{-\omega_1, -\omega_2\} - \left(\|V_{12}\|^2 + \|V_{23}\|^2\right)^{1/2},\end{aligned}$$

where

$$\begin{aligned}p_0 &:= -\frac{1}{6}\left((\omega_1-\omega_2)^2 + \omega_1^2 + \omega_2^2\right) - \|V_{12}\|^2 - \|V_{23}\|^2;\\ q_0 &:= \|V_{23}\|^2\omega_1 - \frac{2}{27}(\omega_1+\omega_2)^3 + \frac{1}{3}\left(\omega_1\omega_2 - \|V_{12}\|^2 - \|V_{23}\|^2\right)(\omega_1+\omega_2);\\ \varphi_0 &:= \arccos\left(-\frac{3q_0}{2p_0}\sqrt{-\frac{3}{p_0}}\right).\end{aligned}$$

Since $\mathbf{H}$ is self-adjoint, both the row and column sum Gershgorin enclosure give the lower bound (see Remark 7.4 (i))

$$\begin{aligned}\min\sigma(\mathbf{H}) \geq g_{\mathrm{row},-}(\mathbf{H}) &= g_{\mathrm{col},-}(\mathbf{H})\\ &= \min\{-\omega_1 - \|V_{12}\|, -\omega_2 - \|V_{12}\| - \|V_{23}\|\},\end{aligned}$$

while classical perturbation theory yields (see Remark 7.4 (ii))

$$\min\sigma(\mathbf{H}) \geq p_-(\mathbf{H}) = \min\{-\omega_1, -\omega_2\} - \|V\|,$$

where $\|V\| = \max_{\|f\|=1}\|Vf\|$.

Finally, the quadratic numerical range $W^2(\mathcal{A})$ with respect to the two different 2×2 operator matrix decompositions

$$\mathbf{H} = \left(\begin{array}{cc|c} -\nabla^2/2+U_1 & V_{12} & 0 \\ \overline{V_{12}} & -\nabla^2/2+U_2 & V_{23} \\ \hline 0 & \overline{V_{23}} & -\nabla^2/2 \end{array}\right) =: \begin{pmatrix} \widetilde{A}_{11} & \widetilde{A}_{12} \\ \widetilde{A}_{21} & \widetilde{A}_{22} \end{pmatrix}, \tag{7.14}$$

$$\mathbf{H} = \left(\begin{array}{c|cc} -\nabla^2/2+U_1 & V_{12} & 0 \\ \hline \overline{V_{12}} & -\nabla^2/2+U_2 & V_{23} \\ 0 & \overline{V_{23}} & -\nabla^2/2 \end{array}\right) =: \begin{pmatrix} \widehat{A}_{11} & \widehat{A}_{12} \\ \widehat{A}_{21} & \widehat{A}_{22} \end{pmatrix}, \tag{7.15}$$

in $L_2(\mathbb{R}^3)^2 \oplus L_2(\mathbb{R}^3)$ and $L_2(\mathbb{R}^3) \oplus L_2(\mathbb{R}^3)^2$, respectively, also provides lower bounds.

Toward this end, in each case, we apply [**13,** Theorem 5.6] twice, first to the 2×2 operator matrices $\widetilde{A}_{11}$ and $\widehat{A}_{22}$ to estimate the minima of their spectra and then to the 2×2 operator matrices in (7.14) and (7.15). If we set

$$\begin{aligned} \delta_{12} &:= \|V_{12}\| \tan\Big(\frac{1}{2}\arctan\frac{2\|V_{12}\|}{|\omega_1-\omega_2|}\Big) \le \|V_{12}\|, \\ \delta_{23} &:= \|V_{23}\| \tan\Big(\frac{1}{2}\arctan\frac{2\|V_{23}\|}{\omega_2}\Big) < \|V_{23}\|, \end{aligned}$$

then [**13,** Theorem 5.6] applied to $\widetilde{A}_{11}$ and to $\widehat{A}_{22}$ yields the estimates

$$\begin{gathered} \min\{-\omega_1,\, -\omega_2\} - \delta_{12} \le \min\sigma(\widetilde{A}_{11}) \le \min\{-\omega_1,\, -\omega_2\}, \\ -\omega_2 - \delta_{23} \le \min\sigma(\widehat{A}_{22}) \le -\omega_2. \end{gathered}$$

If we define

$$\begin{aligned} \widetilde{\delta}_{23} &:= \|V_{23}\| \tan\Big(\frac{1}{2}\arctan\frac{2\|V_{23}\|}{|\min\sigma(\widetilde{A}_{11}) - \min\sigma(\widetilde{A}_{22})|}\Big) \\ &\le \|V_{23}\| \tan\Big(\frac{1}{2}\arctan\frac{2\|V_{23}\|}{\min\{\omega_1,\,\omega_2\}}\Big) < \|V_{23}\|, \\ \widehat{\delta}_{12} &:= \|V_{12}\| \tan\Big(\frac{1}{2}\arctan\frac{2\|V_{12}\|}{|\min\sigma(\widehat{A}_{11}) - \min\sigma(\widehat{A}_{22})|}\Big) \\ &\le \|V_{12}\| \tan\Big(\frac{1}{2}\arctan\frac{2\|V_{12}\|}{\min\{|\omega_2-\omega_1|, |\omega_2-\omega_1+\delta_{23}|\}}\Big) \le \|V_{12}\|, \\ \delta_V &:= \min\{\delta_{12}+\widetilde{\delta}_{23}, \widehat{\delta}_{12}+\delta_{23}\} < \|V_{12}\| + \|V_{23}\|, \end{aligned}$$

then [**13,** Theorem 5.6] applied to the two different 2×2 operator matrices in (7.14) and (7.15) yields the two estimates

$$\min \sigma(\mathbf{H}) \geq \min\{\min \sigma(\widetilde{A}_{11}), 0\} - \widetilde{\delta}_{23} \geq \min\{-\omega_1, -\omega_2\} - \delta_{12} - \widetilde{\delta}_{23},$$
$$\min \sigma(\mathbf{H}) \geq \min\{-\omega_1, \min \sigma(\widehat{A}_{22})\} - \widehat{\delta}_{12} \geq \min\{-\omega_1, -\omega_2 - \delta_{23}\} - \widehat{\delta}_{12}$$

or, combining both bounds,

$$\begin{aligned} &\min\, \sigma(\mathbf{H}) \geq w_-^2(\mathbf{H}) \\ &:= \max\{\min\{-\omega_1, -\omega_2\} - \delta_{12} - \widetilde{\delta}_{23}, \min\{-\omega_1, -\omega_2 - \delta_{23}\} - \widehat{\delta}_{12}\} \\ &\geq \min\{-\omega_1, -\omega_2\} - \delta_V > \min\{-\omega_1, -\omega_2\} - \big(\|V_{12}\| + \|V_{23}\|\big). \end{aligned}$$

Note that, since

$$-\big(\|V_{12}\|^2 + \|V_{23}\|^2\big)^{1/2} \geq -\big(\|V_{12}\| + \|V_{23}\|\big),$$

the lower bound for $w_-^3(\mathbf{H})$ in (7.14) obtained from the cubic numerical range is better than the lower bound for $w_-^2(\mathbf{H})$ in (7.16) obtained from the two possible quadratic numerical ranges.

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